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# Continuum sensitivity analysis and improved Nelson's method for beam shape eigensensitivities

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**Abstract.** Gradient-based optimization techniques require accurate and efficient sensitivity or design derivative analysis. In general, numerical sensitivity methods such as finite differences are easy to implement but imprecise and computationally inefficient. In contrast, analytical sensitivity methods are highly accurate and efficient. Although these methods have been widely evaluated for static problems or dynamic analysis in the time domain, no analytical sensitivity methods have been developed for eigenvalue problems. In this paper, two different analytical methods for shape eigensensitivity analysis have been evaluated: the Continuum Sensitivity Analysis (CSA) and an enhanced version of Nelson's method. They are both analytical techniques but differ in how the analytical differentiation is performed: before and after the discretization, respectively. CSA has been applied to eigenvalue problems for the first time, while Nelson's method has been improved and adapted to shape optimizations. Both methods have been applied to different cases involving shape optimization of beams. Both vibration and buckling problems were analysed considering the eigenvalue as a design variable. Both methods have been successfully applied, and Nelson's method proved to be more convenient for this kind of problem.

## Introduction

Accurate sensitivity analysis is essential to guarantee the convergence of gradient-based optimization techniques. The sensitivity methods can be divided into numerical methods (finite difference, complex step), analytical methods (discrete analytical, continuum), hybrid methods (semi-analytical), or automatic differentiation methods. Analytical methods are preferred over numerical ones because of their higher accuracy and computational efficiency. They do not require convergence studies to find an adequate step size for calculating the numerical derivative, as for the finite difference method and the semi-analytical one [1,2,3]. Furthermore, they do not need the source code of the analysis or to handle complex operations, as needed for automatic differentiation [4,5] or complex step one [6], respectively. Analytical methods offer an accurate and efficient alternative to compute derivatives for structures [7,8,9], fluids [10,11,12], and fluid-structure-interaction problems [13,14] with respect to shape design parameters. Continuum Sensitivity Analysis (CSA) has been developed to compute gradients to be used in shape optimizations for static structural problems or dynamic problems in the time domain. In this work, it has been extended for the first time to eigenvalue problems. On the other hand, Nelson's method has been widely applied for calculating derivatives with respect to design parameters. In this paper, it has been extended for the first time to shape optimizations. Both methodologies are successfully developed and validated in this paper. However, CSA exhibited some limitations in terms of range of applicability and accuracy. Instead, Nelson's method exhibits very good accuracy and computational efficiency even with coarse meshes. In particular, a nonintrusive and element-agnostic approach was pursued for the method to be suitable for standard commercial software in a black box configuration. Because of that, the approach can be adopted for practical applications concerning structural shape optimization.

**Differentiation of the analytical eigenvalue problem**

Considering the structural free vibration or buckling problem, the formal operator equation of the eigenvalue problem can be described by the following general equation:

$$A\boldsymbol{y} = \zeta B\boldsymbol{y}, \boldsymbol{y} \neq 0, \tag{2}$$

where $\boldsymbol{y}$ is used to indicate the eigenfunction, and $\zeta$ is the associated eigenvalue. The following normalization condition has been employed to scale the eigenfunctions:

$$(B\boldsymbol{y}, \boldsymbol{y}) = 1, \tag{3}$$

where $( , )$ indicates the $L_2$-scalar product. The principle of virtual work can be applied to get the variational formulation of the eigenvalue problem. The $L_2$-scalar product on both sides of Eq. (3) with a smooth function $\bar{\boldsymbol{y}}$ satisfying the same boundary conditions as $\boldsymbol{y}$ may be used to obtain the variational equation of the eigenvalue problem as:

$$a(\boldsymbol{y}, \bar{\boldsymbol{y}}) \equiv (A\boldsymbol{y}, \bar{\boldsymbol{y}}) = \zeta(B\boldsymbol{y}, \bar{\boldsymbol{y}}) \equiv \zeta d(\boldsymbol{y}, \bar{\boldsymbol{y}}). \tag{4}$$

A vibrating structure eigenvalue $\zeta_\tau$ on a deformed domain $\Omega_\tau$ is determined by a variational equation of the form:

$$\begin{aligned} a_{\Omega_\tau}(\boldsymbol{y}_\tau, \bar{\boldsymbol{y}}_\tau) &\equiv \iint_{\Omega_\tau} c(\boldsymbol{y}_\tau, \bar{\boldsymbol{y}}_\tau) d\Omega_\tau \\ &= \zeta_\tau \iint_{\Omega_\tau} e(\boldsymbol{y}_\tau, \bar{\boldsymbol{y}}_\tau) d\Omega_\tau \equiv \zeta_\tau d_{\Omega_\tau}(\boldsymbol{y}_\tau, \bar{\boldsymbol{y}}_\tau), \forall \bar{\boldsymbol{y}}_\tau \in Z_\tau, \end{aligned} \tag{5}$$

where $Z_\tau$ is the space of kinematically admissible displacements, and $c( , )$ and $e( , )$ are symmetric bilinear mappings. Since Eq. 5 is homogeneous in eigenfunction $\boldsymbol{y}_\tau$, a normalizing condition must be used to define a unique solution. The following is the one used in this discussion:

$$d_{\Omega_\tau}(\boldsymbol{y}_\tau, \bar{\boldsymbol{y}}_\tau) = 1. \tag{6}$$

The rigorous derivation of such an eigenvalue problem with respect to the shape, due to Choi and Kim [9], brings to the following general formula:

$$\begin{aligned} \zeta' &= 2\iint_\Omega [-c(\boldsymbol{y}, \nabla y\boldsymbol{V}) + \zeta e(\boldsymbol{y}, \nabla \boldsymbol{y})] d\Omega + \int_\Gamma [c(\boldsymbol{y}, \boldsymbol{y}) - \zeta e(\boldsymbol{y}, \boldsymbol{y})] V_n d\Gamma \\ &= 2\iint_\Omega [-c(\boldsymbol{y}, \nabla y\boldsymbol{V}) + \zeta e(\boldsymbol{y}, \nabla \boldsymbol{y}\boldsymbol{V})] d\Omega + \iint_\Omega \operatorname{div}([c(\boldsymbol{y}, \boldsymbol{y}) - \zeta e(\boldsymbol{y}, \boldsymbol{y})]\boldsymbol{V})\, d\Omega. \end{aligned} \tag{7}$$

This formula can be particularized and simplified based on the type of modes and design velocities.

**Differentiation of the numerical eigenvalue problem**

Nelson's method is a discrete analytical sensitivity method and requires the governing equations first, to be discretized and second, to be differentiated. This kind of method involves the derivatives of the Finite Element (FE) matrices. Given the symmetric real matrices $[K]$, $[M]$, $[K']$ and $[M']$ $\in \mathbb{R}^{n\times n}$, where $[K'] \equiv \frac{\partial [K]}{\partial p}$ and $[M'] \equiv \frac{\partial [M]}{\partial p}$ with $p$ shape parameter, let $\lambda \in \mathbb{R}$ and $\{x\} \in \mathbb{R}^n$ solve the following generalized eigenvalue problem:

$$[K]\{x\} = \lambda[M]\{x\}. \tag{8}$$

The numerical eigenvalue is here called $\lambda$ to distinguish it from the analytical one ($\zeta$). Also, the numerical eigenvector is indicated with $\{x\}$ to discern it from the analytical eigenfunction $\boldsymbol{y}$. The numerical eigenvector is assumed to be normalized with respect to the generalized mass:

$$\{x\}^T[M]\{x\} = 1. \tag{9}$$

In a FE structural problem, $[K]$ is the stiffness matrix and $[M]$ can be either of the mass or differential stiffness matrix, based on the type of problem considered. The differentiation of this equation is due to Nelson [15,16] and the following equation is obtained:

$$\lambda' = \{x\}^T([K'] - \lambda[M'])\{x\}. \tag{10}$$

The method requires the derivatives of the stiffness and mass (or differential stiffness) matrices. Because of that, such an approach has not been applied to shape sensitivity problems until now. However, a nonintrusive and element agnostic approach has been developed in this work to calculate the derivative of structural matrices based on the primary analysis matrices and the connectivity of the mesh.

**Applications and results**

Since some sensitivity methods may not work with repeated eigenvalues, both approaches have been here applied to a vibration problem involving repeated eigenvalues. A beam with a circular cross-section ($r = 2.5\ mm$) and a length of $100\ mm$ has been considered. A Simply Supported-Sliding boundary condition has been applied in order to validate the method even when the stiffness matrix is singular. A uniform mesh containing forty beam elements has been created. The length of the beam has been considered as a shape design variable and a uniform design velocity has been employed. The reference values have been found employing the NASTRAN Design Sensitivity and Optimization solution (SOL 200). The comparison of the eigenvalue derivative with respect to the reference one for both CSA and Nelson's method is summarized in Table 1.

*Table 1: First ten elastic natural frequencies and their derivative: comparison between SOL 200, CSA and improved Nelson's method.*

| Mode ID | Natural frequency [Hz] | SOL 200 Derivative [Hz] | CSA Derivative [Hz/mm] | Nelson's derivative [Hz/mm] |
|---|---|---|---|---|
| 1,2 | 308.3 | -6.1685 | -6.1667 (-0.029 %) | -6.1664 (-0.034 %) |
| 3,4 | 998.5 | -19.972 | -19.971 (-0.005 %) | -19.971 (-0.005 %) |
| 5,6 | 2082 | -41.642 | -41.639 (-0.007 %) | -41.640 (-0.005 %) |
| 7,8 | 3558 | -71.162 | -71.156 (-0.008 %) | -71.161 (-0.001 %) |
| 9,10 | 5426 | -108.52 | -108.50 (-0.018 %) | -108.52 ( - ) |

Both methods work with shape sensitivity problems and demonstrate very good accuracy. However, the CSA precision decreases if coarse meshes are employed. This is probably due to the many special gradients necessary for applying Eq. 7. They must be calculated numerically and their accuracy affects the final eigenvalue derivative estimation and also decreases the computational efficiency. Instead, Nelson's method is accurate even with coarse meshes and has a higher computational efficiency.

The two approaches have been then applied to buckling problems. A $1\ m$ long beam with a rectangular cross-section ($8\ mm\ \times 12\ mm$) has been here considered. A uniform mesh with twenty beam elements has been created. Several boundary conditions have been employed for the analysis: Simply Supported - Simply Supported (S-S), Clamped - Clamped (C-C), Clamped - Free

(C-F), Clamped - Guided (C-G), and Simply Supported - Guided (S-G). The buckling eigenvalues along with their derivatives are summarized in Table 2. The CSA and improved Nelson's method have been compared with the analytical results.

*Table 2: Buckling eigenvalues and their derivative with respect to the beam length. CSA and enhanced Nelson's method comparison with analytical results.*

| Boundary Conditions | Eigenvalue [N] | Analytical derivative [N/mm] | CSA derivative [N/mm] | Nelson's derivative [N/mm] |
|---|---|---|---|---|
| S-S | 359.29 | -0.71857 | -0.72480 (0.867 %) | -0.71857 (-) |
| C-C | 1437.16 | -2.87428 | -3.02699 (5.313 %) | -2.87423 (-0.002 %) |
| C-F | 89.821 | -0.17964 | -0.18011 (0.262 %) | -0.17963 (-0.006 %) |
| C-G | 359.29 | -0.71857 | -0.72761 (1.258 %) | -0.71857 (-) |
| S-G | 89.821 | -0.17964 | -0.18011 (0.262 %) | -0.17963 (-0.006 %) |

Nelson's results perfectly match the analytical derivative, while CSA exhibits accuracy limitations even with enough fine meshes. In fact, when particularizing Eq. 7 to buckling problems, even more spatial gradients than vibration problems are required. As a result, Nelson's method is strongly suggested for this kind of application.

**Conclusions**

This work presented alternative methods to calculate shape design derivatives of beam eigenvalue problems. Two innovative solutions have been developed and investigated: the CSA and the enhanced Nelson's method. Both approaches have been successfully applied and validated. However, the CSA exhibited some limitations, especially in the buckling case. Nelson's method, on the contrary, has shown excellent accuracy, and very good computational efficiency. The enhanced Nelson's method can be successfully used in shape sensitivity problems and integrated into design optimization software. Future works will show the application of both approaches to plate and three-dimensional FE models.

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