Modal analysis of hyperelastic structures in non-trivial equilibrium states via higher-order plate finite elements

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Abstract. The present work proposes a higher-order plate finite element model for the three-dimensional modal analysis of hyperelastic structures. Refined higher-order 2D models are defined in the well-established Carrera Unified Formulation (CUF) framework, coupled with the classical hyperelastic constitutive law modeling based on the strain energy function approach. Matrix forms of governing equations for static nonlinear analysis and modal analysis around nontrivial equilibrium conditions are carried out using the Principle of Virtual Displacements (PVD). The primary investigation of the following study is about the natural frequencies and modal shapes exhibited by hyperelastic soft structures subjected to pre-stress conditions.

Introduction

In recent years, renewed interest in soft materials in diverse fields, including mechanical, aeronautical, robotics engineering, and biological applications, has led to the development of new and efficient computational models for numerical simulations. The enhanced elastic properties of soft hyperelastic structures have attracted many researchers, and the dynamic features of hyperelastic materials have garnered increased attention. These features of hyperelastic structures generally lead to highly nonlinear equilibrium equations that do not allow closed-form solutions for static or dynamic problems. Furthermore, classical structural theories for beams and plates have been proven inadequate when considering large strains and nonlinear constitutive law. In this context, the Finite Element Method (FEM) allows a wide range of investigations in terms of material properties, geometries and topology, frequency analysis, and the design phase of components. Exploring the modal behavior of soft hyperelastic structures involves studying pre-stressed conditions and analyzing how large strains or rotations affect natural frequencies and mode shapes. Existing reference solutions in this field typically focus on simple geometries or considered boundary conditions. While the literature has extensively covered various beam-like or plate-like structures, the adoption of classical FEM models, such as hexahedral solid models, is generally associated with an inadequate computational cost required by the numerical simulation. In this scenario, finite element models based on Carrera Unified Formulation (CUF) for the modal analysis of isotropic hyperelastic materials are proposed here. CUF allows, starting from the definition of the displacement field by a recursive index notation, the definition of FE governing equations in terms of invariants to the theory of structure approximation and kinematics assumption [1,2,3]. Refined fully nonlinear structural models are defined then straightforwardly [4]. The capabilities of the proposed plate CUF models are investigated through the static and modal analysis of hyperelastic thin and thick silicon plates, for which mode aberration is observed.

Hyperelastic constitutive modeling

Hyperelastic models adopted in the present work are defined under the classical strain energy function approach based on the Flory decomposition of kinematic measures. The strain energy

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density function, the deformation gradient $\mathbf{F}$ and the right Cauchy-Green strain tensor $\mathbf{C}$ are then written as:

$$\Psi(\mathbf{C}) = \Psi_{\mathrm{vol}}(\mathrm{J}) + \Psi_{\mathrm{iso}}(\bar{\mathbf{C}}) = \mathrm{U}(\mathrm{J}) + \bar{\Psi}(\bar{\mathrm{I}}_1, \bar{\mathrm{I}}_2) \tag{1}$$

$$\mathbf{F} = \mathbf{F}_{vol}\bar{\mathbf{F}} \ \rightarrow\ \mathbf{F}_{vol} = \mathrm{J}^{\frac{1}{3}}\mathbf{I} \quad \bar{\mathbf{F}} = \mathrm{J}^{-\frac{1}{3}}\boldsymbol{F} \tag{2}$$

$$\mathbf{C} = \mathbf{C}_{vol}\bar{\mathbf{C}} \ \rightarrow\ \mathbf{C}_{vol} = \mathrm{J}^{\frac{2}{3}}\mathbf{I} \quad \bar{\mathbf{C}} = \mathrm{J}^{-\frac{2}{3}}\mathbf{C} \tag{3}$$

where $\overline{I_1}, \overline{I_2}$ are the invariants of the isochoric part of the right Cauchy-Green tensor $\bar{\mathbf{C}}$ and J is the volume ratio, the determinant of the deformation gradient. In the present work, the decoupled Mooney-Rivlin model for silicon rubber is taken into account

$$\Psi(\mathbf{C}) = \mathrm{c}_{10}(\bar{\mathrm{I}}_1 - 3) + \mathrm{c}_{01}(\bar{\mathrm{I}}_2 - 3) + \frac{1}{\mathrm{D}_1}(\mathrm{J}-1)^2 \tag{4}$$

where $D_1 = 2/k$ is the incompressibility parameter defined from the bulk modulus k. Stress measure represented by the second Piola-Kirchoff stress tensor (PK2) is defined in its decoupled form:

$$\mathbf{S} = \frac{\partial\Psi}{\partial\mathbf{C}} = \mathbf{S}_{vol} + \mathbf{S}_{iso} = \mathrm{J}p\mathbf{C}^{-1} + \mathrm{J}^{-\frac{2}{3}}\left(\mathbf{I} - \frac{1}{3}\mathbf{C}^{-1}\otimes\mathbf{C}\right) : \bar{\mathbf{S}} = \mathrm{J}p\mathbf{C}^{-1} + \mathrm{J}^{-\frac{2}{3}}\,\mathbf{P}:\bar{\mathbf{S}} \tag{5}$$

where $\boldsymbol{P}$ is the fourth-order projection tensor adopted in the Total Lagrangian Formulation finite element formulation, p is the hydrostatic pressure and $\bar{\mathbf{S}}$ is the rescaled/modified PK2 stress tensor. Due to the presence of both material and geometrical nonlinearities, an incremental formulation is here adopted. Following the procedure by Holzapfel [5], the constitutive equation Eq. (5) is rewritten in its differential form:

$$\Delta\mathbf{S} = \mathcal{C}\frac{1}{2}\Delta\mathbf{C} = \mathcal{C}\Delta\mathbf{E} \tag{6}$$

where $\mathcal{C}$ is the so-called tangent elasticity tensor, defined starting the linearization of the constitutive law. The complete derivation of the explicit expression of the tangent elasticity tensor (or material Jacobian tensor) can be found again in [5].

Higher-order structural theories

The Unified Formulation for static and modal analysis of hyperelastic soft structures has been already proposed in [6], in which hyperelastic higher-order beam finite element models are established. In the following, modal analysis of compressible hyperelastic plate structures are performed adopting higher-order plate (2D) models. In CUF, the three-dimensional displacement field is expressed as a polynomial expansion of the generalized nodal displacements, coupling approximation expansion theories along the thickness with kinematic models along the plate mid-surface:

$$\boldsymbol{u}(x,y,z) = F_\tau(z)\boldsymbol{u}_\tau(x,y) = F_\tau(z)N_i(x,y)\boldsymbol{u}_{\tau i} \quad \tau = 1,\dots,\mathrm{M}, \quad \mathrm{i} = 1,\dots,\mathrm{N_n} \tag{8}$$

where M is the related to the order of the structural theory adopted, $\boldsymbol{F_\tau(z)}$ is the set of expansion functions, representing the theory of structure approximation, $\boldsymbol{N_i(x,y)}$ is the set of 2D shape functions of the discrete N_n finite nodes along the mid-surface, and finally $\mathbf{u}_{\boldsymbol{\tau}\mathbf{i}}$ is the vector of generalized displacement component.. In the present work, Lagrange Expansion (LE) class are considered, starting from the set of Lagrange's polynomials [2].

Governing equations

The equilibrium equations for the static and undamped vibration problem are carried out by means of the Principle of Virtual Displacements (PVD), written as:

$$\delta\mathcal{L}_{ine} + \delta\mathcal{L}_{int} = \delta\mathcal{L}_{ext} \tag{9}$$

where $\boldsymbol{\mathcal{L}_{ine}}$ is the work done by inertial forces, $\boldsymbol{\mathcal{L}_{int}}$ and $\boldsymbol{\mathcal{L}_{ext}}$ represents work done by internal and external forces, and $\boldsymbol{\delta}$ denotes the virtual variation. Adopting the same indices notation introduced for the displacement field also for the Green-Lagrange strain tensor and the virtual quantities, one can derive the FN (Fundamental Nuclei) of the internal and external forces vector and mass matrix, obtaining the matrix form of the governing equation.

$$\delta u_{sj}:\ \boldsymbol{M}^{\tau sij}\ddot{u}_{\tau i} + \boldsymbol{F}_{int}^{sj} = \boldsymbol{F}_{ext}^{sj} \ \rightarrow\ \boldsymbol{M}\ddot{\boldsymbol{u}} + \boldsymbol{F}_{int} = \boldsymbol{F}_{ext} \tag{9}$$

The nonlinear governing equation is then linearized to implement an incremental numerical solver based on path-following constraints. Through a Taylor expansion truncated at the first order, one can derive the incremental equation:

$$\boldsymbol{M}\Delta\ddot{\boldsymbol{u}} + \boldsymbol{K}_T\Delta\boldsymbol{u} = -\boldsymbol{\phi}_{res}(\boldsymbol{u}_0, \ddot{\boldsymbol{u}}_0, \boldsymbol{p}_0) + \boldsymbol{I}\Delta\lambda\boldsymbol{p}_{ref} \quad (9)$$

Finally, imposing a harmonic solution of the type $\boldsymbol{\Delta u} = \boldsymbol{\Phi e^{i\omega t}}$, the classical linear eigenvalue problem is then obtained, that gives the natural frequencies and the normal modes of vibration around the computed non-trivial equilibrium state:

$$(\boldsymbol{K}_T - \omega^2\boldsymbol{M})\boldsymbol{\Phi} = 0 \quad (9)$$

More detail about the derivation of the undamped vibration problem around non-trivial equilibrium conditions of hyperelastic soft structures can be found in [7], where the FN of the tangent stiffness matrix and the complete linearization procedure is detailed.

Numerical results

The investigated case study involves a clamped square plate of compressible hyperelastic material. Small amplitude vibrations around non-trivial equilibrium conditions are explored considering a thick square plate with a lateral side of a=b=1 and h=0.1 m. In the further investigations, the material density of the hyperelastic beam is set to a typical value for silicone rubber $\rho = 1200\ \mathrm{kg/m^3}$. The fitted material parameters of the Mooney-Rivlin model for silicon rubber are fixed to $c_{10} = 0.14\ \mathrm{MPa}$ and $c_{01} = 0.023\ \mathrm{MPa}$ [7], and Poisson's ratio $\nu = 0.4$. The mathematical models utilized different LE models along the plate thickness, namely the LE2 parabolic and LE3 cubic expansion models, and various finite element discretizations around the plate mid-surface to assess the accuracy and efficiency of the proposed model. The relative percentage difference and the computational costs in terms of DOF (degrees of freedom) will be further presented. The numerical results obtained via higher-order 2D CUF plate elements are compared with the 3D solution carried out through ABAQUS commercial software. First, a free vibration analysis around the undeformed equilibrium condition is performed to analyze the natural frequencies and modal shapes of the thick plate. Table 1 shows the first five natural frequencies, comparing the results with the solution obtained with the 3D model in ABAQUS. The most accurate model involves 20x20 Q9 elements along the mid-surface, which will be further adopted as discretization. Furthermore, the nonlinear static analysis is performed, considering a uniform pressure at the top surface to establish nontrivial equilibrium conditions. Figure (1) illustrates the equilibrium curve obtained using higher-order 2D CUF elements and the 3D ABAQUS solution. Around the marked nontrivial equilibrium conditions, the undamped vibration problem is solved, computing the natural frequencies for increasing applied pressure values. Figure (2) presents the first eight natural frequencies and their variations with increasing applied pressure, with results compared to the proposed 3D solution. Accurate results are consistently achieved, and mode aberration is observed.

Conclusions

This manuscript discusses the undamped vibration problem around non-trivial equilibrium conditions of hyperelastic plate structures. First, soft hyperelastic plates have been modeled using higher-order CUF-based finite elements and have proven to guarantee accurate results regarding displacements and modal behavior of thin and thick structures. Mode aberration is observed, such as crossing, as observed in the proposed study, for specific critical values of the applied load. Future works will investigate hyperelastic multilayered soft plates and shells, the modal analysis of bio-inspired structures, and the effect of fiber reinforcement.

Table 1: Cantilever compressible silicon thick plate: free vibration problem around the undeformed condition, convergence analysis on the first five natural frequencies [Hz]. Comparison between 2D CUF results and 3D ABAQUS results.

Mesh	Mode 1	Mode2	Mode 3	Mode 4	Mode 5	DOFs
10x10 Q9	$0.4421^{0.417\%}$	$1.0421^{0.809\%}$	$2.6030^{0.876\%}$	$2.7948^{0.156\%}$	$3.2900^{0.629\%}$	3969
12x12 Q9	$0.4418^{0.003\%}$	$1.0411^{0.708\%}$	$2.6002^{0.768\%}$	$2.7937^{0.115\%}$	$3.2880^{0.568\%}$	5625
15x15 Q9	$0.4415^{0.271\%}$	$1.0403^{0.626\%}$	$2.5979^{0.682\%}$	$2.7927^{0.079\%}$	$3.2866^{0.524\%}$	8649
20x20 Q9	$0.4412^{0.211\%}$	$1.0396^{0.565\%}$	$2.5962^{0..616\%}$	$2.7919^{0.051\%}$	$3.2856^{0.495\%}$	15129
ABQ 3D 12500 C3D20R	0.4403	1.0338	2.5804	2.7905	3.2695	177633

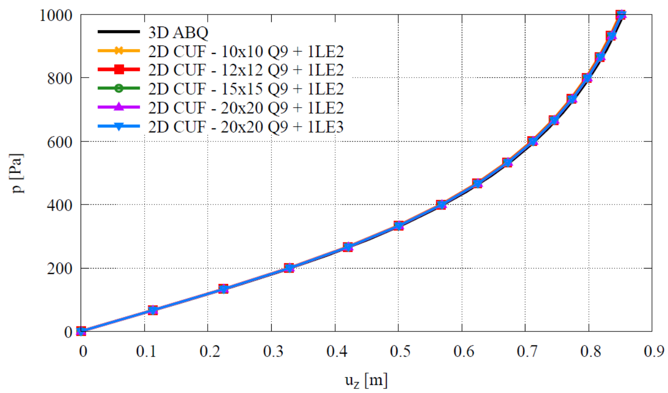
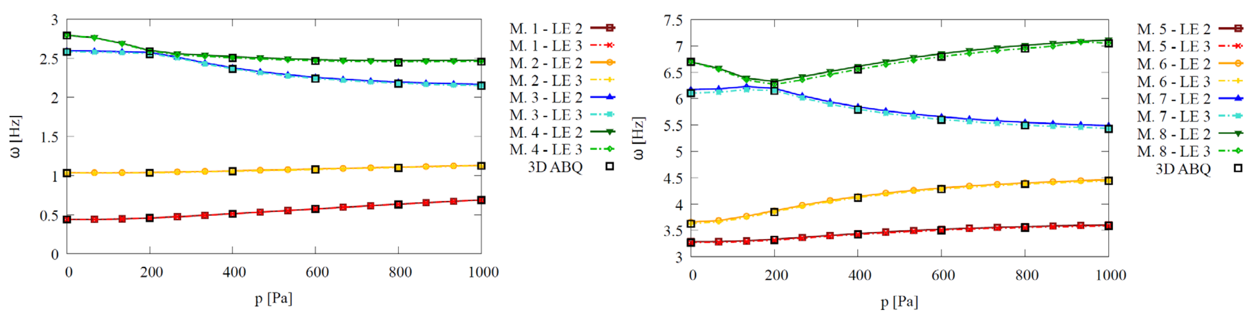


Figure 1: Cantilever compressible silicon thick plate: equilibrium paths



(a) Modes 1-4 *(b) Modes 5-8*

Figure 2: Cantilever compressible silicon plate, case L/h = 10: variation of the first eight natural frequencies for increasing value of the applied pressure.

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