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Selection of best beam theories based on natural frequencies and dynamic response obtained through mode superposition method

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Abstract. Simplified formulations, particularly 1D models, are fundamental for reducing the computational cost typically required by structural analyses. The use of a limited number of nodal degrees of freedom has inevitable implications for the model's capabilities and accuracy. Furthermore, the performance of a reduced formulation is strictly problem-dependent, and the choice of a specific set of primary unknowns must be weighted considering their influence on the accuracy of the results of interest. In this work, a procedure for the selection of the best 1D models to adopt for time-response analyses is investigated. Through the Axiomatic/Asymptotic Method (AAM), the influence of single unknowns is evaluated for a specific structural configuration, which can be described as a combination of aspect ratio, material, geometry, and boundary conditions. The finite element governing equations for every considered set of variables are obtained through the Carrera Unified Formulation (CUF). The main indicator for the quality of a theory is based on the evaluation of a certain number of natural frequencies. Dynamic response analyses are then carried out using the modal superposition method to further asses the performance of the selected best theories.

Introduction

The development of accurate reduced 1D models is a crucial topic in structural mechanics, and it is primarily tied to the need for computational cost reduction. Many efforts were made over the years to improve their capabilities and reduce the gap in accuracy with complete 3D formulations, resulting in a wide variety of approaches [1-4]. Among them, the adoption of higher-order polynomial expansions to describe the displacement field above the cross-section proved remarkably successful, allowing for a proper but still efficient modeling of more complex mechanical behaviors [5].

However, the performances of higher-order theories (HOT) are strictly problem-dependent, and a further increase in the order of the expansion would result in undesired growth of computational demand. These aspects highlight the need for the definition of a theory selection approach able to optimize the number of variables to include in a model, identifying the most influential ones to opt for the derivation of accurate results concerning a specific application.

In this direction, a powerful tool to guide the modeling process is the Axiomatic/Asymptotic Method [6-8]. By directly comparing the results stemming from different combinations of expansion terms and a reference solution, the AAM can inform about the achievable accuracy given the level of complexity and related computational cost of a specific theory. AAM-like procedures require many results to be compared. These can be conveniently obtained in the framework of the Carrera Unified Formulation [9], which provides a generalized methodology for the implementation of structural theories of any type and order, independently from the considered structural problem.

Finite Element Formulation
For beam-like structures, the finite element models are built using the reference system presented in Fig. 1, with the x-z plane laying on the cross-section, referred to as **Ω**.

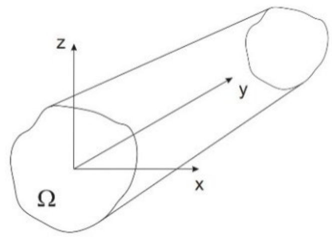

Figure 1 - Reference coordinate system for beam model

The displacement field can be expressed as:

$$\mathbf{u}(x,y,z) = \{u_x, u_y, u_z\}^T \quad (1)$$

In the framework of the Carrera Unified Formulation, 1D models can be refined to provide a better description of the cross-sectional mechanical behavior. The displacement field on the cross-section at a specific y coordinate can be modeled by introducing expansion functions F_τ and F_s:

$$\mathbf{u}(x,y,z) = F_\tau(x,z)\mathbf{u}_\tau, \qquad \delta\mathbf{u}(x,y,z) = F_s(x,z)\delta\mathbf{u}_s \quad (2)$$

$$\tau, s = 1, \ldots, M$$

The Einstein notation acts on subscripts τ and s. M denotes the order or number of terms of the expansion, δ is the variational operator used to express the virtual variations. Here, $\mathbf{u}_\tau$ represents the generalized displacement variables involved in the expansion functions ($\delta\mathbf{u}_s$ being their virtual variations). In this work, only expansion functions based on Taylor polynomials were used.

The Finite Element discretization over the beam axis can be obtained by introducing the shape functions $N_{i,j}$. The previous equations thus become:

$$\mathbf{u}(x,y,z) = N_i(y)F_\tau(x,z)\mathbf{q}_{\tau i}, \quad \delta\mathbf{u}(x,y,z) = N_j(y)F_s(x,z)\delta\mathbf{q}_{sj} \quad (3)$$

$$\tau, s = 1, \ldots, M \qquad i, j = 1, \ldots, N_n$$

where N_n is the number of FE nodes, $\mathbf{q}_{\tau i}$ and $\delta\mathbf{q}_{sj}$ are the vectors of nodal unknown variables. For a full beam model of order M=2, the first of Eqs. 3 can be written in extended form as:

$$\begin{aligned} u_x &= u_{x1} + xu_{x2} + zu_{x3} + x^2u_{x4} + xzu_{x5} + z^2u_{x6} \\ u_y &= u_{y1} + xu_{y2} + zu_{y3} + x^2u_{y4} + xzu_{y5} + z^2u_{y6} \\ u_z &= u_{z1} + xu_{z2} + zu_{z3} + x^2u_{z4} + xzu_{z5} + z^2u_{z6} \end{aligned} \quad (4)$$

with six nodal displacement variables for each component, resulting in a total of eighteen.

The Principle of Virtual Displacements (PVD) is used to derive the governing differential equations. The complete formulation of the dynamic problem can be obtained through a hierarchical and generalized assembly procedure [9] over all nodes, resulting in:

$$\mathbf{M}\ddot{\mathbf{q}}(\mathrm{t}) + \mathbf{K}\mathbf{q}(\mathrm{t}) = \mathbf{P}(\mathrm{t}) \tag{5}$$

The undamped dynamic problem can be solved using the mode superposition method [10, 11], which is more computationally efficient than direct integration schemes such as the Newmark method. The mode superposition method involves the transformation of Eq. 5 into modal coordinates. The finite element nodal point displacements are then obtained by superposition of the response in each mode.

Axiomatic/Asymptotic Method

Techniques like the Axiomatic/Asymptotic Method can be used to assess the impact of individual expansion terms on the model accuracy. By directly comparing the solutions obtained from all possible combinations of generalized unknown variables, the AAM can highlight the most influential terms given a particular structural configuration.

The preliminary step of an Axiomatic/Asymptotic procedure is the selection of a reference solution, typically provided by a full high-order expansion or 3D formulation.

The insights offered by the AAM can be conveniently summarized through the Best Theory Diagram, a graphical representation of the relationship between the number of adopted unknowns and achievable accuracy.

Preliminary Results

The proposed methodology is here introduced considering a simply-supported multi-bay box beam [12]. The sides of the section are b = 0.38 m and h = 0.14 m, with thickness of the wall t = 0.02 m and length L = 10b. An isotropic material was considered, with E=75 GPa, ν=0.33, and density ρ = 2700 kg/m^3. The beam was discretized using 10 B4 elements, and only theories up to the fourth order were considered. The full fourth-order expansion (E4) was also used as the reference model. Considering that the E4 model has 45 nodal unknowns, 2^{45} theories should have been compared.

To reduce the required results, the corresponding expansion terms in all three displacement components were activated/deactivated together, thus requiring only 2^{15} total computations. Furthermore, the constant and linear terms of the expansion were always kept active, for a final total of 2^{12} compared theories.

The selection of the best theories was performed using the average error over the first thirty natural frequencies:

$$\%\mathrm{E}_{\mathrm{AVG}} = \frac{1}{30} \cdot \sum_{\mathrm{i}=1}^{30} 100 \cdot \frac{\left|\mathrm{f_i} - \mathrm{f_i^{E4}}\right|}{\mathrm{f_i^{E4}}} \tag{6}$$

Table 1 provides the resulting best-performing models. A black triangle signals an active term of expansion. Follows the corresponding Best Theory Diagram in Fig. 2.

Each determined best theory can then be tested through the comparison of the dynamic response to a time-varying load. For the presented case, two out-of-phase sinusoidal loads were applied at points P_1 (b/2-t/2, L/2, h/2) and P_2 (-b/2+t/2, L/2, -h/2), both with an amplitude F_0 = 10000 N and ω = 5 rad/s. The vertical displacements over time at P_1 obtained with some of the found Best Theories are compared in Fig. 3. The provided results demonstrate the weight that different expansion terms have on the solution accuracy for this specific structural case and the computational advantage that model optimization can guarantee.

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Table 1 - Best Theories for simply-supported multi-bay box beam based on the average percentage error over the first thirty natural frequencies.

Nodal DOFs	const.	x	z	x^2	xz	z^2	x^3	x^2z	xz^2	z^3	x^4	x^3z	x^2z^2	xz^3	z^4	$\%E_{AVG}$
45	▲	▲	▲	▲	▲	▲	▲	▲	▲	▲	▲	▲	▲	▲	▲	0.0000
42	▲	▲	▲	▲	▲	▲	▲	▲	▲	▲	▲	▲	△	▲	▲	0.0263
39	▲	▲	▲	▲	▲	▲	▲	▲	▲	▲	△	▲	△	▲	▲	0.0329
36	▲	▲	▲	▲	▲	▲	▲	▲	▲	▲	△	▲	△	▲	△	0.0803
33	▲	▲	▲	▲	▲	▲	▲	▲	△	▲	△	▲	△	▲	△	0.1478
30	▲	▲	▲	▲	▲	▲	▲	△	△	▲	△	▲	△	▲	△	0.4045
27	▲	▲	▲	▲	▲	▲	▲	△	△	▲	△	△	△	▲	△	0.8190
24	▲	▲	▲	▲	▲	▲	△	△	△	▲	△	△	△	▲	△	1.3431
21	▲	▲	▲	▲	▲	▲	△	△	△	▲	△	△	△	△	△	2.8689
18	▲	▲	▲	▲	▲	△	△	△	△	▲	△	△	△	△	△	4.7301
15	▲	▲	▲	▲	▲	△	△	△	△	△	△	△	△	△	△	7.1115
12	▲	▲	▲	△	▲	△	△	△	△	△	△	△	△	△	△	10.6242
9	▲	▲	▲	△	△	△	△	△	△	△	△	△	△	△	△	22.9546

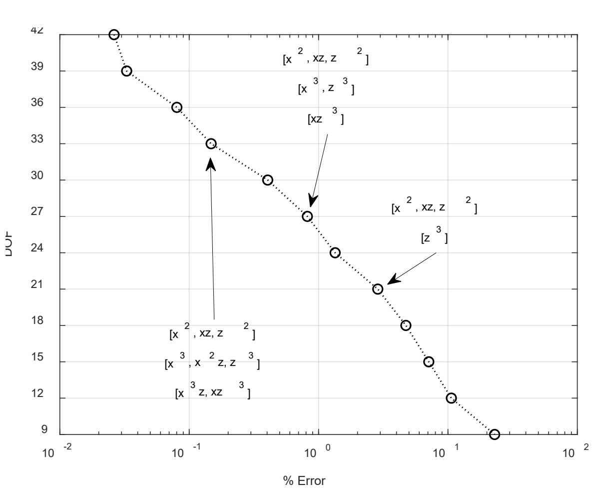

Figure 2 – Best Theory Diagram for simply-supported multi-bay box beam based on the average percentage error over the first thirty natural frequencies.

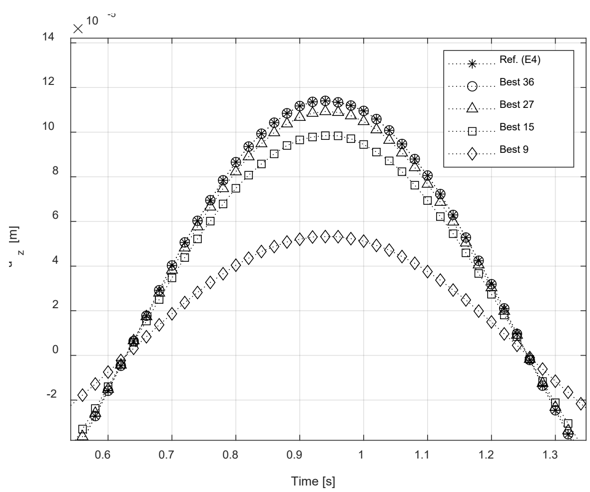

Figure 3 – Time response for simply - supported multi-bay box beam, u_z evaluated in P_1.

Conclusions

This work investigates a methodology for selecting the best beam models for a specific structural configuration.

The Axiomatic/Asymptotic Method is employed to derive the best theories due to the marked problem dependence of the models' accuracy.

Free-vibration and time response analyses are considered, and a performance indicator based on the quality of the estimated natural frequencies is adopted. The study highlights the influence of specific expansion terms on the provided solutions' accuracy and the selection criterion's influence on the outcome of the AAM.

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