Tensor as a tool in engineering analysis

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Abstract. This paper underscores the potency of the invariant character of tensor and its derivative concepts and accentuate the synergy between isotropic tensor and other tensors and the corresponding vector operations. The equivalence of *covariant derivative* in a curvilinear coordinates system embedded with a non-constant vector field and the *partial derivative* in an affine coordinates system ingrained with a constant vector field is interrogated. The corresponding role of the Christofell symbols as the affine connector of vectors with their derivatives in a variable field are compared to the *Frenet-Seret* skew-matrix connecting the *trihedrons* (i.e. *tangent*, *normal* and *binormal*) of a moving space curve with their derivatives. The nexus of the Christofell symbols with the *geodesics* is also shown. The structure of the metric tensor $\tilde{I}$ and the Levi-Chivita skew-symmetric tensor $\tilde{\boldsymbol{\varepsilon}}$, as isotropic tensor rank-2 and rank-3 respectively is highlighted, such that the usual operations of *dot product* (or *scalar product* or *inner product*) and *cross product* or (*vector product* or *spin/rotation operation*}) are now expressed through the isotropic tensors. Recalling the theory of *exterior differential form* and invoking the Poincare's theorem we show the application of the exterior product in establishing *exact differential* (or *total differential*) in calculus in relation to *plane problem* of Elasticity. The invariant nature of the tensor objects and operations therefrom are then copiously invoked and deployed to establish constitutive relation for materials: in finite elasticity, within the context of *hyperelasticity*; composites, where there is a trade-off between heterogeneity and anisotropy through homogenisation process whereby differential equations with variable coefficients are converted to differential equations with constant coefficients; and plasticity, where application of tensor is exhibited with strain gradient plasticity, and shown how the concepts provide balance of microscopic forces, balance of macroscopic forces, and plastic flow laws as concise mathematical equations.

Introduction

This work, *Tensor as a Tool in Engineering Analysis* provides a brief survey of what tensor is, its relevance and application to continuum mechanics and engineering investigations and analysis. *Tensor theory*, simply and succinctly put, is the ***the theory of invariants***; which encompasses familiar physical objects such as *scalars, vectors, stress, strain*, *isotropes* and similar objects of various ranks and pertinent operations on them. It is even *a shorthand tool* in underpinning or driving home fundamental physical concepts, concisely [1-3]. The richness in the invariant nature of tensor, for example, enables us realize the equivalence in the actions of *covariant derivative* operator in a varying field, with curvilinear coordinates system, and the *partial derivative* operator in *affine*/rectilinear coordinates system. What is more, the effect of covariant derivative of metric tensor corresponds to the partial derivative of the Kronecker delta tensor, as a vanishing quantity.

Tensor as a discipline provides a potent and convenient tool to interrogate and navigate the complex labyrinth of the world of continuum mechanics and by implication, the modeling of real life phenomenon and engineering designs and analysis [4-5]. So, it is often pertinent and profitable to possess good and deep knowledge and understanding of this tool and its deployment in

scientific, engineering and technological endeavours. This, we would demonstrate in a number of papers that would be presented in this session of the Conference.
In the setting of engineering mathematics/continuum mechanics as related to engineering design, underlining governing rules are deduced from concepts similar to those used in theoretical mechanics or dynamics [6]. Fundamental amongst these principles are the: *conservation of mass, conservation of linear momentum, conservation of moment of linear momentum, conservation of energy*. These constitute the so-called conservation laws. They are complimented by the so-called constitutive laws, in solving problems for specific material of a continuum; be it elastic, plastic, viscoelastic, viscoplastic, fliud - linear or nonlinear [7-8].

Here, a formal definition of tensor is given, and the relevance of the Christofell symbols as a coefficient of affine connectivity or coefficient of proportionality of a vector to its derivative in a varying space in analogy to the Frenet-Seret Matrix for a space curve is highlighted; it also serves as a measure of geodesics in an Euclidean space. The theory of exterior form is invoked, through the Poincare's Theorem for a differential form, to establish the Airy's stress function for plane elasticity [9-12]. The concept of *Homogenisation* [13-15] for periodic composites is illuminated, giving its implication as a process converting a system of partial differential equations with variable coefficients to that of constant coefficients, but with an incurred anisotropy as penalty. The property of isotropic tensor is explored to navigate the process of tensor derivative of functions of tensor argument. The deep endowment of tensor operations is richly displayed with the interrogation of strain-gradient-divergence plasticity [16-17].

Tensor

An object $\tilde{\mathbf{T}} \in \mathsf{T} \subset \Omega(\mathsf{E}^n)$ is a tensor in an n-dimensional sub-space of Euclidean space E^n if it is invariant under the transformation of coordinates system $\mathbf{q}(q^i)$, however, such that the components vary linearly and homogeneously, under the said transformation:
$\mathsf{O}: q^i \rightarrow q'^i,\ i = 1, 2, 3, ..., n$.

Let $\tilde{\mathbf{T}}$ be a tensor in a three-dimensional euclidean space E^n in which we have introduced an arbitrary curvilinear system of coordinates $\mathbf{q} = (q^1, q^2, q^3, \ldots)$ with the accompanying orthogonal covariant basis $\mathbf{e}_1, \mathbf{e}_2, \mathbf{e}_3, \ldots$ and the corresponding contravariant basis $\mathbf{e}^1, \mathbf{e}^2, \mathbf{e}^3, \ldots$. Then, either through the contravariant base vectors (basis) $\mathbf{e}^i$ or the covariant base vectors (basis) $\mathbf{e}_i$ we can express an *n-ranked* tensor $\tilde{\mathbf{T}}$ as

$$\tilde{\mathbf{T}} = T^{i_1 i_2 \cdots i_n} \mathbf{e}_{i_1} \mathbf{e}_{i_2} \cdots \mathbf{e}_{i_n} \text{ or } \tilde{\mathbf{T}} = T_{i_1 i_2 \cdots i_n} \mathbf{e}^{i_1} \mathbf{e}^{i_2} \cdots \mathbf{e}^{i_n}. \tag{1.1.1}$$

Now, if in E^n in place of $\mathbf{q}$ we introduce another coordinate system $\mathbf{q}'$, such that these two systems relate to one another by the non-singular matrix of transformation (or the so-called fundamental matrix)

$$\frac{\partial q^i}{\partial q^{i'}} \quad \text{and its inverse} \quad \frac{\partial q^{i'}}{\partial q^i}; \quad \left| \frac{\partial q^i}{\partial q^{i'}} \right| \neq 0, \tag{1.1.2}$$

then, the invariant character of $\tilde{\mathbf{T}}$ is expressed in the following.

(i) $\tilde{\mathbf{T}}'(\mathbf{q}') = T^{i'_1 i'_2 \cdots i'_n}(\mathbf{q}')\mathbf{e}_{i'_1}\mathbf{e}_{i'_2} \cdots \mathbf{e}_{i'_n} = T_{i'_1 i'_2 \cdots i'_n}(\mathbf{q}')\mathbf{e}^{i'_1}\mathbf{e}_{i'2} \cdots \mathbf{e}^{i'_n}$

$$= T_{i_1 i_2 \cdots i_n}(\mathbf{q})\mathbf{e}^{i_1}\mathbf{e}^{i_2} \cdots \mathbf{e}^{i_n} = T^{i_1 i_2 \cdots i_n}(\mathbf{q})\mathbf{e}_{i_1}\mathbf{e}_{i_2} \cdots \mathbf{e}_{i_n} = \tilde{\mathbf{T}}(\mathbf{q}). \tag{1.1.3}$$

(ii) But, with respect to the components, they transform accordingly and we have

$$T_{i'_1 i'_2 \cdots i'_n}(\mathbf{q}') = T_{i_1 i_2 \cdots i_n}(\mathbf{q}) \frac{\partial q^{i_1}}{\partial q^{i'_1}} \frac{\partial q^{i_2}}{\partial q^{i'_2}} \cdots \frac{\partial q^{i_n}}{\partial q^{i'_n}},\ T^{i'_1 i'_2 \cdots i'_n}(\mathbf{q}') = T(\mathbf{q})^{i_1 i_2 \cdots i_n} \frac{\partial q^{i'_1}}{\partial q^{i_1}} \frac{\partial q^{i'_2}}{\partial q^{i_2}} \cdots \frac{\partial q^{i'_n}}{\partial q^{i_n}}; \tag{1.1.4}$$

$$T_{i_1i_2\cdots i_n}(\mathbf{q}) = T_{i_1'i_2'\cdots i_n'}(\mathbf{q}')\frac{\partial q^{i_1'}}{\partial q^{i_1}}\frac{\partial q^{i_2'}}{\partial q^{i_2}}\cdots\frac{\partial q^{i_n'}}{\partial q^{i_n}}\quad T^{i_1i_2\cdots i_n}(\mathbf{q}) = T^{i_1'i_2'\cdots i_n'}(\mathbf{q}')\frac{\partial q^{i_1}}{\partial q^{i_1'}}\frac{\partial q^{i_2}}{\partial q^{i_2'}}\cdots\frac{\partial q^{i_n}}{\partial q^{i_n'}}. \quad (1.1.4)'$$

Here, $T^{i_1i_2\cdots i_n}(\mathbf{q})$ and $T^{i_1'i_2'\cdots i_n'}(\mathbf{q}')$ are the *contravariant components* of the tensor $\tilde{\mathbf{T}}$ in the coordinates systems $\mathbf{q}$ and $\mathbf{q}'$ respectively. Similarly, $T_{i_1i_2\cdots i_n}(\mathbf{q})$ and $T_{i_1'i_2'\cdots i_n'}(\mathbf{q}')$ are referred to as the *covariant component*s of the tensor $\tilde{\mathbf{T}}$ in the coordinates systems $\mathbf{q}$ and $\mathbf{q}'$ respectively. For example, g_{ij}, g^{ij} the *covariant metric tensor*, *contravariant metric tensor* are respectively $g_{ij} \equiv \mathbf{e}_i \cdot \mathbf{e}_j$ and $g^{ij} \equiv \mathbf{e}^i \cdot \mathbf{e}^j$.

Christofell Symbols and Relevance

The objects Γ_{ijk} and Γ^k_{ij} are known as the *Christofell symbols* of *first kind* and *second kind* or simply, *1st kind* and *2nd kind* respectively. The 2nd kind is defined as

$$\frac{\partial \mathbf{e}_m}{\partial q^i} \equiv \Gamma^k_{im}\mathbf{e}_k,\ \frac{\partial \mathbf{e}^m}{\partial q^i} \equiv -\Gamma^m_{ik}\mathbf{e}^k;\ \Gamma_{ijm} = g_{mk}\Gamma^k_{ij},\ \Gamma^m_{ij} = g^{mk}\Gamma_{ijk},\ i,j,k,m = 1,2,3. \quad (1.2.1)$$

The Christofell symbols, related to one another as above, and to the metric tensor through the expression:

$$\Gamma_{ijk} = \frac{1}{2}(g_{kj,i} + g_{ik,j} - g_{ij,k}),\ \Gamma^m_{ij} = \frac{1}{2}g^{mk}(g_{kj,i} + g_{ik,j} - g_{ij,k}),\ i,j,k,m = 1,2,3. \quad (1.2.2)$$

They constitute what is called <u>*Symbols of Space Connectivity*</u> or <u>*Coefficients of Affine Connectivity*</u>.

Frenet-Seret Matrix analogous to Christofell Symbols

It could be noted that in this, the *2nd kind Christofell Symbols* plays an anologuous role as the *Frenet-Seret matrix* in the case of space curves, where the matrix relates the non-constant unit orthogonal *thriedral* ($\mathbf{t}, \mathbf{n}, \mathbf{b}$) (i.e. *tangent*, *normal* and *bi-normal*) to their derivatives ($\frac{d\mathbf{t}}{ds}, \frac{d\mathbf{n}}{ds}, \frac{d\mathbf{b}}{ds}$), on the space curve, parametrised by s, the *arclength*; $\mathbf{r} = \mathbf{r}(s)$:

$$\begin{pmatrix} \frac{d\mathbf{t}}{ds} \\ \frac{d\mathbf{n}}{ds} \\ \frac{d\mathbf{b}}{ds} \end{pmatrix} = \begin{pmatrix} 0 & \kappa & 0 \\ -\kappa & 0 & \tau \\ 0 & -\tau & 0 \end{pmatrix}\begin{pmatrix} \mathbf{t} \\ \mathbf{n} \\ \mathbf{b} \end{pmatrix}. \text{Frenet-Seret } \textit{matrix} \text{ and } \textit{rule} \quad (1.2.3)$$

The covariant derivative

The covariant derivative of any mixed tensor $T^{i_1i_2\cdots i_m}_{j_1j_2\cdots j_n}$ is denoted and given by the expression

$$\nabla_k T^{i_1i_2\cdots i_m}_{j_1j_2\cdots j_n} \equiv \frac{\partial}{\partial q^k}T^{i_1i_2\cdots i_m}_{j_1j_2\cdots j_n} - T^{i_1i_2\cdots i_m}_{pj_2\cdots j_n}\Gamma^p_{j_1k} - T^{i_1i_2\cdots i_m}_{j_1pj_3\cdots j_n}\Gamma^p_{j_2k}\cdots - T^{i_1i_2\cdots i_m}_{j_1j_2\cdots j_{n-2}pj_n}\Gamma^p_{j_{n-1}k} - T^{i_1i_2\cdots i_m}_{j_1j_2\cdots j_{n-1}p}\Gamma^p_{j_nk}$$

$$+T^{pi_2\cdots i_m}_{j_1j_2\cdots j_n}\Gamma^{i_1}_{pk} + T^{i_1pi_3\cdots i_m}_{j_1j_2\cdots j_n}\Gamma^{i_2}_{pk}\cdots + T^{i_1i_2\cdots i_{m-2}pi_m}_{j_1j_2\cdots j_n}\Gamma^{i_{m-1}}_{pk} + T^{i_1i_2\cdots i_{m-2}i_{m-1}p}_{j_1j_2\cdots j_n}\Gamma^{i_m}_{pk}. \quad (1.2.4)$$

We emphasize, the partial derivative of the Kronecker delta tensor is to the covariant derivative of the metric tensor as the rectilinear coordinates (affine coordinates) system is to the curvilinear coordinates system. In fact, the partial derivative of Kronecker delta δ_{ij} and the covariant derivative of the metric tensor g_{ij} correspondingly vanish:

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$$\partial\delta_{ij} = 0 \to \nabla_k g_{ij} = \frac{\partial g_{ij}}{\partial q^k} - g_{mj}\Gamma^m_{ki} - g_{im}\Gamma^m_{jk} = \frac{\partial g_{ij}}{\partial q^k} - \Gamma_{ki,j} - \Gamma_{jk,i} \overset{(1.2.2)}{=} 0. \quad (1.2.4)'$$

Measure of Geodesics

The concept of *Geodesics* derives from the notion of *absolute derivative*. In fact, let a space curve be parametrized on interval $I \subset \mathrm{R}$ with parameter $t \in I$, such that its position vector is

$$\mathbf{r} = \mathbf{r}(u^1(t), u^2(t), u^3(t)) = \mathbf{r}(t). \quad (1.2.5)$$

Note that

$$\mathbf{e}_i = \mathbf{e}_i(u^k(t)) \equiv \frac{\partial \mathbf{r}}{\partial u^i}; \ \frac{d\mathbf{e}_i(u^k(t))}{dt} = \frac{\partial \mathbf{e}_i}{\partial u^k}\frac{du^k}{dt} \overset{(1.2.1)}{=} \Gamma^m_{ik}\mathbf{e}_m \frac{du^k}{dt}. \quad (1.2.6)$$

Now, we can consider the derivative of a vector $\mathbf{v} = v^i\mathbf{e}_i$ along this curve in the domain with arbitrary coordinates u^i and the orthogonal basis $\mathbf{e}_i$. Its derivative with respect to parameter t, taking cognizance of the basis $\mathbf{e}_i$ not being constant, in a fashion similar to the process of covariant derivative, is

$$\frac{d\mathbf{v}}{dt} = \frac{dv^i}{dt}\mathbf{e}_i + v^i\frac{d\mathbf{e}_i}{dt} = \frac{dv^i}{dt}\mathbf{e}_i + v^i\Gamma^m_{ij}\frac{dv^j}{dt}\mathbf{e}_m = \left(\frac{dv^i}{dt} + v^m\Gamma^i_{mj}\frac{dv^j}{dt}\right)\mathbf{e}_i; \ \nabla_t \equiv \frac{dv^i}{dt} + v^m\Gamma^i_{mj}\frac{dv^j}{dt}, \quad (1.2.7)$$

where $\nabla_t \equiv \nabla_i$ is referred to as the *absolute derivative* of the component v^i of vector $\mathbf{v}$. This is at times also referred to as *intrinsic derivative*. Note that this notion can be extended to invariants of higher rank.

We can similarly deduce the absolute derivative of the (i) covariant vector $v_i(t)$, (ii) covariant tensor $T_{ij}(t)$, (iii) mixed tensor T_i^j and contravariant tensor T^{ij}.

A geodesic is *a curve* $\mathbf{r}$ = $\mathbf{r}(s)$ *parametrized with the arclength* s *such that the absolute derivative of its tangent vanishes*. (i.e. *a curve of constant tangent* or *vanishing acceleration*). This is given by the expression

$$\frac{d\mathbf{t}}{ds} = 0 \text{ i.e.} \frac{dt^i}{ds} + \Gamma^i_{mj}t^m\frac{du^j}{ds} = 0, \to \frac{d^2u^i}{ds^2} + \Gamma^i_{mj}\frac{du^j}{ds}\frac{du^m}{ds} = 0, \quad (1.2.8)$$

noting that $t^i = du^i / ds$, $\mathbf{t} \equiv \dfrac{d\mathbf{r}}{ds}$.

Example

The Gauss curvature and the geodesic line for the Poincare's half plane,

$$ds^2 = \frac{du^2 + dv^2}{v^2}, \ v > 0 \quad (1.2.9)$$

are obtained respectively as

$$K = -\frac{R_{1212}}{g} = \frac{b_{11}b_{22} - b_{12}^2}{g_{11}g_{22} - g_{12}^2} = -1; \ (u-u_0)^2 + v^2 = c^2.$$

Indeed, taking into recognition (1.2.8) and (1.2.9), the required geodesic line is the family of curves $(u-u_0)^2 + v^2 = c^2$ (i.e. shifted circles); noting that for the half plane (1.2.9),

$$g_{11} = \frac{1}{v^2}, g_{12} = g_{21} = 0, \ g_{22} = \frac{1}{v^2} \ \ g^{11} = v^2, g^{12} = g^{21} = 0, \ g^{22} = v^2.$$

$$\Gamma^1_{11} = 0, \ \Gamma^1_{12} = -\frac{1}{v}, \ \Gamma^1_{22} = 0, \ \Gamma^2_{11} = \frac{1}{v}, \ \Gamma^2_{12} = 0, \ \Gamma^2_{22} = -\frac{1}{v}.$$

The Gauss curvature is deduced from the expression relating the Riemann curvature R_{1212} and the Gauss curvature K such that $R_{1212} = -gK = b_{11}b_{22} - b_{12}^2$, while the determinant of the 1st fundamental matrix of the surface is $g = g_{11}g_{22} - g_{12}^2$ and the determinant of the 2nd fundamental matrix of a surface $b_{11}b_{22} - b_{12}^2$ is $-(\frac{1}{v^2})^2$..

Exterior Form and Application

Definition of Exterior Form

Consider Q, the space of m-$tuple$ tensor product of the conjugate space $\Re^*$ on itself. That is $Q = \Re^* \otimes \Re^* \otimes \cdots \otimes \Re^* \otimes \Re^*$. Let Q_o be a space of skew-symmetric/anti-symmetric tensor, a subspace of Q; $Q_o \subset Q$. Then, the element $\Phi \in Q_o$ is called the *Exterior Form*, rank-m and presented as

$$\Phi = \varphi_{i_1 i_2 \ldots i_{m-1} i_m} \mathbf{e}^{i_1} \Lambda \mathbf{e}^{i_2} \cdots \Lambda \mathbf{e}^{i_{m-1}} \Lambda \mathbf{e}^{i_m}, \ \backslash\ m = 1,2,3,\ldots, \tag{2.1.1}$$

where $\mathbf{e}^{i_k}, k = 1,2,3,\ldots$ are referred to as *covectors.*

Differential Form

An *Exterior Differential Form* ω is any exterior form constructed on differential basis; when the covectors are now the *differential*, dx^{i_k}.

$$\omega = a_{i_1 i_2 \ldots i_{m-1} i_m}(x^1, x^2, \ldots, x^{n-1}, x^n) dx^{i_1} \Lambda dx^{i_2} \ldots \Lambda dx^{i_{m-1}} \Lambda dx^{i_m}, \tag{2.1.2}$$

where in any $n-$dimensional domain Ω, the ring of coefficients

$$a(x^1, x^2, \ldots, x^{n-1}, x^n), \ a \in C^q(\Omega); \ q \geq 1.$$

i.e. a is $q-$times continuously differentiable or possesses partial derivatives of order q inclusively.

The base elements (covectors) are the differentials of the variables $x^1, x^2, \ldots, x^{n-1}, x^n$, $dx^1, dx^2, \ldots, dx^{n-1}, dx^n$ and view them as abstract unit. It is obvious that all the properties of an exterior form are endowed any differential forms [11,18].

The Exterior differentiation of an exterior differential form (2.1.2) is defined as

$$\begin{aligned} \mathsf{D}\omega &\equiv da_{i_1 i_2 \ldots i_{m-1} i_m}(x^1, x^2, \ldots, x^{n-1}, x^n) dx^{i_1} \Lambda dx^{i_2} \ldots \Lambda dx^{i_{m-1}} \Lambda dx^{i_m} \\ &= \frac{\partial a_{i_1 i_2 \ldots i_{m-1} i_m}}{\partial x^i} dx^j \Lambda dx^{i_1} \Lambda dx^{i_2} \ldots \Lambda dx^{i_{m-1}} \Lambda dx^{i_m}, \ i = 1,2,3,\ldots, n-1, n. \end{aligned} \tag{2.1.3}$$

We note that *The operation of exterior diferential form (like that of covariant differentiation) increases the rank of a form by one*. And when a change of coordinates is executed we have

$$dx^{i_k} \to dx^{i'_k} \to dx^{i'_1} \Lambda dx^{i'_2} \ldots \Lambda dx^{i'_{n-1}} \Lambda dx^{i'_n} = det \left| \frac{\partial x^{i'}}{\partial x^i} \right| dx^{i_1} \Lambda dx^{i_2} \ldots \Lambda dx^{i_{n-1}} \Lambda dx^{i_n}.$$

Poincare Theorem

The second exterior differential of an exterior form is zero,

$$\mathsf{DD}\omega = 0. \tag{2.2.1}$$

Proof:

The second exterior differential is

$$\mathsf{DD}\omega \overset{(2.1.3)}{=} \frac{\partial^2 a_{i_1 i_2 \ldots i_{m-1} i_m}}{\partial x^j \partial x^k} dx^k \Lambda dx^j \Lambda dx^{i_1} \Lambda dx^{i_2} \ldots \Lambda dx^{i_{m-1}} \Lambda dx^{i_m}.$$

$$\text{But } a_{i_1 \ldots i_m}(x^1, \ldots, x^n) \in C^m(\Omega) \to \frac{\partial^2 a_{i_1 i_2 \ldots i_{m-1} i_m}}{\partial x^j \partial x^k} = \frac{\partial^2 a_{i_1 i_2 \ldots i_{m-1} i_m}}{\partial x^k \partial x^j}.$$

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At the same time, the form $dx^k \Lambda dx^j \Lambda dx^{i_1} \Lambda dx^{i_2} \ldots \Lambda dx^{i_{m-1}} \Lambda dx^{i_m}$ is skew-symmetric, including in the indices k and j. For this, we have $-\mathsf{DD}\omega = \mathsf{DD}\omega$. Hence, the required proof, $\mathsf{DD}\omega = 0$.

Example

Let a linear differential form be $\omega = Pdx + Qdy$. By (2.1.3), the exterior differential form is

$$\mathsf{D}\omega \overset{(2.1.3)}{=} dP\Lambda dx + dQ\Lambda dy \overset{(2.1.3)}{=} \frac{\partial P}{\partial y} dy\Lambda dx + \frac{\partial Q}{\partial x} dx\Lambda dy = (\frac{\partial Q}{\partial x} - \frac{\partial P}{\partial y}) dx\Lambda dy; \;\; dx\Lambda dy = -dx\Lambda dy. \quad (2.2.2)$$

We note that (i.) ω is a total differential if $\frac{\partial P}{\partial y} = \frac{\partial Q}{\partial x}$, since $d\omega = \frac{\partial \omega}{\partial x} dx + \frac{\partial \omega}{\partial y} dy$ and $\frac{\partial^2 \omega}{\partial x \partial y} = \frac{\partial^2 \omega}{\partial y \partial x}$. (ii.) if ω is a total differential then, $\frac{\partial P}{\partial y} = \frac{\partial Q}{\partial x}$ and we shall obtain from (2.2.2) that $\mathsf{D}\omega = 0$. And by the corollary to the Poincare's theorem, ω is a differential form. (iii.) the concept of differential form can be used to establish whether or not an expression constitutes a *total differential*.

Application of Poincare's Theorem in Elasticity

The boundary value problem of *plane elasticity* can be formulated in terms of the Airy's function, $\varphi(x_1, x_2)$. At the root of this formulation is the Poincare's theorem, deriving from the theory of *Exterior Form* or *Exterior Differential Form*. This fundamental background, often, is omitted in textbooks of elasticity.

Theorem: $\frac{\partial \sigma_{11}}{\partial x_1} + \frac{\partial \sigma_{12}}{\partial x_2} = 0; \frac{\partial \sigma_{12}}{\partial x_1} + \frac{\partial \sigma_{22}}{\partial x_2} = 0$ if only and only if $\sigma_{11} dx_2 - \sigma_{12} dx_1$ and $\sigma_{22} dx_1 - \sigma_{12} dx_2$ are respectively an exact differentials.

Proof: By Poincare's theorem, ω is an exact differential if $d\omega = 0$. That is, we can find φ such that $\omega = d\varphi$ and consequently, $\omega = dd\varphi = 0$. Thus, $\sigma_{11} dx_2 - \sigma_{12} dx_1$ is an exact differential $\rightarrow$ $d(\sigma_{11} dx_2 - \sigma_{12} dx_1) = 0$. As a differential form, we have

$$\begin{aligned} d(\sigma_{11} dx_2 - \sigma_{12} dx_1) &= \frac{\partial}{\partial x_1}(\sigma_{11} dx_2 - \sigma_{12} dx_1) dx_1 + \frac{\partial}{\partial x_2}(\sigma_{11} dx_2 - \sigma_{12} dx_1) dx_2 \\ &= \frac{\partial \sigma_{11}}{\partial x_1} dx_2 \Lambda dx_1 - \frac{\partial \sigma_{12}}{\partial x_1} dx_1 \Lambda dx_1 + \frac{\partial \sigma_{11}}{\partial x_2} dx_2 \Lambda dx_2 - \frac{\partial \sigma_{12}}{\partial x_2} dx_1 \Lambda dx_1 \\ &= (\frac{\partial \sigma_{11}}{\partial x_1} + \frac{\partial \sigma_{12}}{\partial x_2}) dx_2 \Lambda dx_1 = 0; \text{ since } dx_1 \Lambda dx_1 = dx_2 \Lambda dx_2 = 0. \Rightarrow (\frac{\partial \sigma_{11}}{\partial x_1} + \frac{\partial \sigma_{12}}{\partial x_2}) = 0. \end{aligned}$$

Conversely, Let $(\frac{\partial \sigma_{11}}{\partial x_1} + \frac{\partial \sigma_{12}}{\partial x_2}) = 0 \Rightarrow d(\sigma_{11} dx_2 - \sigma_{12} dx_1) = 0$. Then, $\sigma_{11} dx_2 - \sigma_{12} dx_1$ is an exact differential. $\Rightarrow \exists$ a function $P(x_1, x_2)$;

$$\sigma_{11} dx_2 - \sigma_{12} dx_1 = dP(x_1, x_2) = \frac{\partial P}{\partial x_1} dx_1 + \frac{\partial P}{\partial x_2} dx_2; \Rightarrow \sigma_{11} = \frac{\partial P}{\partial x_2}, \sigma_{12} = -\frac{\partial P}{\partial x_1}. \quad (*)$$

Similarly: There exists a function $Q(x_1, x_2)$;

$$\sigma_{22} dx_1 - \sigma_{12} dx_2 = dQ(x_1, x_2) = \frac{\partial Q}{\partial x_1} dx_1 + \frac{\partial Q}{\partial x_2} dx_2; \Rightarrow \sigma_{22} = \frac{\partial Q}{\partial x_i}, \sigma_{12} = -\frac{\partial Q}{\partial x_2}. \quad (**)$$

By (*) and (**) $\Rightarrow \exists$ the *form* $\varphi(x_1, x_2)$ (i.e. *Airy's stress function*) such that

$$\sigma_{11} = \frac{\partial^2 \varphi}{\partial x_2^2}, \sigma_{22} = \frac{\partial^2 \varphi}{\partial x_1^2}, \sigma_{12} = \frac{\partial^2 \varphi}{\partial x_2 \partial x_1} = \frac{\partial^2 \varphi}{\partial x_1 \partial x_2} = \sigma_{21}.$$

Homogenisation of Heterogeneous Medium

Linear Elasticity: A Sample Elastostatic Problem

Consider, in a three-dimensional Euclidean space E^3, the already familiar boundary value problem of classical elastostatics in terms of displacement $\mathbf{u} \in \Omega \subset \mathsf{E}^3$,

$$[C_{ijkl}u_{k,l}]_{,j} + \rho X_i = 0, \text{ in } \quad \Omega, \tag{3.1}$$

$$\varepsilon_{ij} = \frac{1}{2}(u_{i,j} + u_{j,i}), \text{ in } \quad \Omega, \tag{3.2}$$

$$\sigma_{ij} = C_{ijkl}\varepsilon_{kl}, \text{ in } \quad \Omega, \tag{3.3}$$

$$u_i |_{\Sigma_u} = u_i^0; \quad C_{ijkl}u_{k,l}n_j |_{\Sigma_\sigma} = S_i^0, \text{ on } \quad \Sigma = \Sigma_u \bigcup \Sigma_\sigma, \tag{3.4}$$

where C_{ijkl} are the generalized Hooke's material characteristics/constants, X_i is the mass or volume body force, u_i is the displacement, σ_{ij} is the stress, ρ is the density, ε_{ij} is the strain, u_i^0 is the displacement specified on $\Sigma_u = \partial\Omega_u$, a part of the boundary of Ω, while S_i^0 is the load/force specified on $\Sigma_\sigma = \partial\Omega_\sigma$, the other part of the boundary Ω: $\Sigma_u \bigcup \Sigma_\sigma = \partial\Omega_u \bigcup \partial\Omega_\sigma = \partial\Omega$. Here and elsewhere, $f_{m,n} \equiv \dfrac{\partial f_m}{\partial x_n}$, $i,j,k,l = 1,2,3$.

Alternate Compatibility Equation - Chezaro's Formular:

Taking advantage of tensor notation and its invariance property, an alternate presentation of the compatibility equation (or Cauchy equations) (3.2) is provided by Chezaro [9-10]. This is just the solution to the differential equations (3.2); obtained through integration of (3.2):

$$u_i = u_i^0 + \omega_{ij}^0(x'_j - x_j^0) + \int_M^{M'} [\varepsilon_{ij} + (x'_n - x_n)(\frac{\partial \varepsilon_{in}}{\partial x_j} - \frac{\partial \varepsilon_{jn}}{\partial x_i})]dx_j, \tag{3.2*}$$

where the body has been fixed at point M^0, with displacement u_i^0 and rotation tensor

$\omega_{i,j}^0 = \dfrac{1}{2}(u_{i,j}^0 - u_{j,i}^0)$. Now, let Ω be an heterogeneous medium, say a *lamina composite*, of isotropic elastic layers.

The tensor rank-4 (i.e. the elastic characteristics) C_{ijkl} will be non-constant, but a periodic function of the coordinates

$$C_{ijkl}(\mathbf{x}) = C_{ijkl}(\mathbf{x}, \mathbf{l}), \tag{3.5}$$

where l is the periodic geometric length. We introduce second argument (or the so-called *local coordinate* or the *fast coordinate*) ξ in which case $\mathbf{x}$ is referred to as the *global coordinate*, and then take the asymptotic expansion [13,15]

$$u_i(\mathbf{x}, \boldsymbol{\xi}) = \sum_{m=0} \alpha^m N_{ipq_1\ldots q_m}^{(m)}(\xi) v_{p,q_1\ldots q_m}(\mathbf{x}); \quad \xi = \frac{\mathbf{x}}{\alpha}, \quad |\alpha| < 1, \tag{3.6}$$

where $\alpha \equiv \dfrac{l}{L}$ is the so-called *small parameter*, which is the ratio of characteristic lengths l and L of the periodic cell ω and the whole body Ω respectively.

We then insert (3.6) in (3.1), (3.2) to obtain ensuing boundary value problems, now for the *homogenised* medium, howbeit now with acquired anisotropy:

$$\sum_{m=0} \alpha^m h_{ipq_1\ldots q_{m+2}}^{(m)} v_{p,q_1\ldots q_{m+1}}(\mathbf{x}) + X_i = 0, \tag{3.1'}$$

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$$\sum_{m=0} \alpha^m N^{(m)}_{ipq_1\ldots q_m}(\xi) v_{p,q_1\ldots q_m}(\mathbf{x})|_{\partial\Omega_1} = u_i^0; \ \sum_{m=0} \alpha^m h^{(m)}_{ipq_1\ldots q_{m+2}} v_{p,q_1\ldots q_{m+1}}(\mathbf{x}) n_{q_{m+2}} |_{\partial\Omega_1} = S_i^0, \qquad (3.2)'$$

where

$$\mathbf{N^m} = \mathbf{N^{(m)}_{ipq_1\ldots q_m}}(\xi) \text{ and } \mathbf{h^m} = \mathbf{h^{(m)}_{ipq_1\ldots q_{m+2}}}$$

are the periodic *structural/local function* and
the *effective material elastic constants*, the latter which is independent of both $\mathbf{x}$ and ξ.

$$\mathbf{v}(\mathbf{x}) \equiv \mathbf{v_p}(\mathbf{x}) \equiv < \mathbf{u_0} >$$

is the *average* displacement vector, independent of ξ,

$$N^p \equiv \delta_{ij}, \ i,j = 1,2,3,\ldots; N^m = 0 \text{ if } m<0, \ \delta_{ij} = \begin{cases} 1 & \text{if } \ i=j, \\ 0 & \text{if } \ i \neq j, \end{cases} i,j = 1,2,3, \qquad (3.7)$$

where δ_{ij} is the so-called *Kronecker delta*, unit tensor.

Mechanical Characteristics $\tilde{h}$ and Conjugate Local Functions $\mathbf{N}(\xi)$

To obtain $\tilde{h}$, the mechanical characteristics, and the corresponding $\tilde{N}(\xi)$, the *structural functions* (or *local functions*) we solve the so-called *non-boundary valued periodic problems* $P^{(m+1,m)}$, sequentially:

$$h^{(m)}_{ipq_1\ldots q_{m+2}} = (C_{ijkl} N^{(m+2)}_{ipq_1\ldots q_{m+2}}(\xi)_{|_l})_{|_j} + (C_{ijq_{m+1}k} N^{(m+2)}_{ipq_1\ldots q_{m+1}}(\xi))_{|_j}$$
$$+ C_{iq_{m+2}kl} N^{(m+1)}_{kpq_1\ldots q_{m+1}}(\xi)_{|_l} + C_{iq_{m+2}kq_{m+1}} N^{(m)}_{kpq_1\ldots q_m}(\xi)_{|_l}, \ m = -1,0,1,\ldots$$
$$h^{(m)}_{ipq_1\ldots q_{m+2}} = \langle C_{iq_{m+2}kl} N^{(m+1)}_{kpq_1\ldots q_{m+1}}(\xi)_{|_l} + C_{iq_{m+2}kq_{m+1}} N^{(m)}_{kpq_1\ldots q_m}(\xi) \rangle, \qquad (3.9)$$

where for any function $f(\mathbf{x},\xi) \in \Omega$, $f(\mathbf{x},\xi)_{,m} \equiv \dfrac{\partial f}{\partial x_m}$, while $f(\mathbf{x},\xi)|_m \equiv \dfrac{\partial f}{\partial \xi_m}$.

Thus, at the zeroth level, we have the periodic problem $P^{(1,0)}$. This implies that the boundary value problem (3.1)-(3.4), which is a set of equations with variable coefficients, has now become a system of differential equations now with constant coefficients; howbeit, with an incurred anisotropy:

$$[h_{ijkl} v_{k,l}]_{,j} + \rho X_i = 0, \text{ in } \Omega, \qquad (3.1)''$$

$$\varepsilon^{(0)}_{ij} = \frac{1}{2}(v_{i,j} + v_{j,i}), \text{ in } \Omega, \qquad (3.2)''$$

$$\sigma_{ij} = h_{ijkl} \varepsilon^{(0)}_{kl}, \text{ in } \Omega, \qquad (3.3)''$$

$$v_i|_{\Sigma_u} = u_i^0; \ \ h_{ijkl} v_{k,l} n_j|_{\Sigma_\sigma} = S_i^0, \text{on } \ \Sigma = \Sigma_u \bigcup \Sigma_\sigma, \qquad (3.4)''$$

Isotropic Tensor and Invariant Operations

Isotropy

We recall that *isotropy* is the property of a material such that every direction in it constitutes an axis of rotational symmetry.

Isotropic Scalar Function:

A *scalar function* $\phi(a^{ij})$ of a tensor $\mathbf{A} = (a^{ij})$ is isotropic if it preserves its value and form of dependence on those components in any orthogonal mapping H of the coordinates system

$$\phi(a'^{ij}) = \phi(a^{ij}), \ \mathsf{H} : a^{ij} \to a'^{ij}, \ \mathbf{r}' = \mathbf{r} \cdot \mathsf{H}.$$

Isotropic Tensor:
A tensor is *isotropic* if its components are invariant under any orthogonal mapping of the base vectors in which it is defined. (That is, both the tensor and its components are invariant.)
Note: Any would-be tensor, by definition, is meant to be invariant under mappings of coordinates system, while only its components would admit appropriate changes by the imposed rule of transformation. But for an isotropic tensor, even its components are required to be unchanged.
Using these properties of isotropic function and tensor, energy functions can be constructed and continuum mechanics problems solved [10,12].

Example
Let $\tilde{O}$ be an orthogonal tensor that maps base vector $\mathbf{e}_i$ into another base vector $\mathbf{e}'_i = \mathbf{e}_i \cdot \tilde{O}^t$. Consider the unit metric tensor $\tilde{I} = g_{ij}\mathbf{e}^i\mathbf{e}^j$, which is known to be an isotropic tensor. Under transformation of its components we have

$$g_{ij} = \mathbf{e}_i \cdot \mathbf{e}_j = \mathbf{e}'_i \cdot \tilde{O} \cdot \tilde{O}^t \cdot \mathbf{e}'_j = \mathbf{e}'_i \cdot \mathbf{e}'_j = g'_{ij} \Rightarrow \tilde{I} = g_{ij}\mathbf{e}^i\mathbf{e}^j = g^{ij}\mathbf{e}_i\mathbf{e}_j = \mathbf{e}^i\mathbf{e}_i = \mathbf{e}_i\mathbf{e}^i$$

is isotropic.

Basic Isotropic Tensor
The basic isotropic tensors include [10]:
(i.) The *rank-0 isotropic tensor*, (the *scalar*), (ii.) The *rank-2 isotropic tensor*, (the unit tensor $\tilde{I}$); it is the only rank-*2* isotropic tensor, and (iii.) The *rank-3 isotropic tensor*, (the Levi-Chivita tensor õ).
Any other isotropic rank-3 tensor $\tilde{\mathbf{T}}$ at most is a constant multiple of it, λõ, where λ is a constant number (hence, õ is *pseudoisotropic tensor*):

$$\text{õ} = -\tilde{I} \times \tilde{I};\ \text{ò}^{ijk}\text{ò}_{lmn} = \begin{vmatrix} \delta_{il} & \delta_{im} & \delta_{in} \\ \delta_{jl} & \delta_{jm} & \delta_{jn} \\ \delta_{kl} & \delta_{km} & \delta_{kn} \end{vmatrix};\delta_{ij} = \begin{cases} 1 & \text{if} \quad i = j, \\ 0 & \text{if} \quad i \neq j, \end{cases} i,j,k,m,n = 1,2,3, \tag{4.1.1}$$

where δ_{ij} is the so-called *Kronecker delta*.

(iv.) The rank-4 isotropic tensor ${}^4\tilde{C}$ consists of the basic isomers

$$\tilde{C}_I \equiv \tilde{I}\tilde{I} = \mathbf{e}_i\mathbf{e}^i\mathbf{e}_j\mathbf{e}^j,\ \tilde{C}_{II} \equiv \mathbf{e}_i\mathbf{e}_j\mathbf{e}^i\mathbf{e}^j,\ \tilde{C}_{III} \equiv \mathbf{e}_i\tilde{I}\mathbf{e}^i = \mathbf{e}_i\mathbf{e}_j\mathbf{e}^j\mathbf{e}^i. \tag{4.1.2}$$

Isotropic Tensor in Product Operation with Rank-2 Tensor
$\forall$ vectors $\mathbf{a},\mathbf{b},\mathbf{c}$ and rank-2 tensors $\tilde{P},\tilde{Q}$ we highlight interaction of the isotropic tensors on them via indicated operations.

$$\tilde{I} \cdot\cdot \mathbf{ab} = \mathbf{a}\cdot\mathbf{b} = a_i b_j \delta_{ij}, \text{õ} \cdot\cdot \mathbf{ab} = \mathbf{a} \times \mathbf{b} = \text{ò}^{ijk} a_i b_j \mathbf{e}_k = \mathbf{c},$$

$$\text{õ}\cdot\text{õ} = (\tilde{I} \times \tilde{I}) \cdot (\tilde{I} \times \tilde{I}) = \tilde{I} \times \tilde{I} \times \tilde{I} = \tilde{C}_{II} - \tilde{C}_{III} <-> (\tilde{C}_{III} - \tilde{C}_{II}) \cdot\cdot \tilde{\mathbf{T}} = -(\tilde{\mathbf{T}} - \tilde{\mathbf{T}}^t), \tag{4.1.3}$$

$$\text{õ} \cdot\cdot \text{õ} = \mathbf{e}_s\mathbf{e}_k \cdot \mathbf{e}^s\mathbf{e}^k - \mathbf{e}_s\mathbf{e}_m \cdot \mathbf{e}^m\mathbf{e}^s = \tilde{I} - 3\tilde{I} = -2\tilde{I},$$

$$\tilde{C}_I \cdot \tilde{P} = \tilde{I} \cdot \tilde{P};\ \ \tilde{C}_{III} \cdot \tilde{P} = \tilde{P} \cdot \tilde{C}_{III}; \tilde{C}_I \cdot\cdot \tilde{P} = I_1(\tilde{P})\tilde{I} = \tilde{P} \cdot\cdot \tilde{C}_I; \tilde{C}_{II} \cdot\cdot \tilde{P} = \tilde{P} \cdot\cdot \tilde{C}_{II} = \tilde{P}^t;\ \ \tilde{C}_{III} \cdot\cdot \tilde{P} = \tilde{P} \cdot\cdot \tilde{C}_{III} = \tilde{P},$$

$$(\tilde{C}_I \cdot\cdot \tilde{P}) \cdot \tilde{Q} = I_1(\tilde{P})\tilde{Q};\ (\tilde{C}_{II} \cdot\cdot \tilde{P})\tilde{Q} = \tilde{P}^t \cdot \tilde{Q};\ \ (\tilde{C}_{III} \cdot\cdot \tilde{P}) \cdot \tilde{Q} = \tilde{P} \cdot \tilde{Q}, \tag{4.1.3'}$$

$$\tilde{C}_I \cdot\cdot \tilde{P} \cdot \tilde{Q} = I_1(\tilde{P} \cdot \tilde{Q})\tilde{I};\ \ \tilde{C}_{II} \cdot\cdot \tilde{P} \cdot \tilde{Q} = \tilde{P}^t \cdot \tilde{Q}^t;\ \ \tilde{C}_{III} \cdot\cdot \tilde{P} \cdot \tilde{Q} = \tilde{P} \cdot \tilde{Q}.$$

Lemma

The space of *rank*- 4 isotropic tensors ${}^4\tilde{C}$ is spanned by its *isomers*, $\{\tilde{C}_I, \tilde{C}_{II}, \tilde{C}_{III}\}$.

Proof

Since the set $\{\tilde{C}_I, \tilde{C}_{II}, \tilde{C}_{III}\}$ (4.1.2), forms the basis for the rank-4 isotropic tensor, then given constants λ, μ, ν we have:

$$ {}^4\tilde{C} = \lambda\tilde{C}_I + \mu(\tilde{C}_{III} + \tilde{C}_{II}) + \nu(\tilde{C}_{III} - \tilde{C}_{II}). \tag{4.1.4}$$

It could be seen that each of the tensors $\tilde{C}_\alpha, \alpha{=}I, II, III$ is isotropic. In fact, for $\tilde{C}_{III}$, $\mathbf{e}'^i = \mathbf{e}^i \cdot O^k_{.m}\mathbf{e}_k\mathbf{e}^m = O^i_{.m}\mathbf{e}^m, \mathbf{e}'_i = \mathbf{e}_i \cdot O^{.n}_m\mathbf{e}^m\mathbf{e}_n = O^{.n}_i\mathbf{e}_n$. Then, for $\tilde{C}_{III}$, $\tilde{C}'_{III} = \mathbf{e}'^i\tilde{I}\mathbf{e}'_i = O^i_{.m}O^{.n}_i\mathbf{e}^m\tilde{I}\mathbf{e}_n = \delta^n_m\mathbf{e}^m\tilde{I}\mathbf{e}_n = \mathbf{e}^n\tilde{I}\mathbf{e}_n = \tilde{C}_{III}$. This confirms isotropy of $\tilde{C}_{III}$, which is similarly true for the other two isomers. Further, it would be seen that $\tilde{C}_\alpha \cdot\cdot \tilde{C}_\beta = \tilde{C}_\gamma, \alpha, \beta, \gamma = I, II, III$. Hence, (4.1.4).

Remark

[a] Any isotropic tensor *rank*- n ($n > 2$) is expressed through the rank- 2 isotropic tensor, $\tilde{I}$.
[b] We further note (i.) that, in consonance with the concept of *isometry*, there could be other ways of writing the basic isotropic tensors; (ii.) any n-ranked tensor has $n!$ isomers. $\tilde{\mathbf{T}}^t$ is the isomer of $\tilde{\mathbf{T}}$. Likewise, $C_{ijk}\mathbf{e}^i\mathbf{e}^j\mathbf{e}^k$ has one of its isomers as $C_{ijk}\mathbf{e}^k\mathbf{e}^j\mathbf{e}^i$.; (iv.) the structure of a given tensor determines the number of independent isomers it will possess, such that the existence of symmetry in its internal structure reduces the number of independent isomers.
For this, a rank-4 tensor will possess $4! = 24$ isomers. But if this tensor is isotropic, then only 3 of its isomers are independent. Hence, for any constants λ, μ and ν any rank-4 isotropic tensor is expressed through the 3 isomers:
[c] In the orthonormal system of coordinates, (4.1.4) reduces to the known *Hooke's elastic tensor* in the case of isotropic material. It is constructed on the rank-4 isotropic tensor $\delta_{ij}\delta_{kl}$, which has the three *isomers* ($\delta_{ij}\delta_{kl}$, $\delta_{ik}\delta_{jl}$, $\delta_{il}\delta_{jk}$), for which the elastic tensor C_{ijkl} for isotropic material takes the expression $C_{ijkl} = \lambda\delta_{ij}\delta_{kl} + \mu(\delta_{ik}\delta_{jl} + \delta_{il}\delta_{jk})$, λ and μ are the Lame constants.

Isotropy

Further, given any rank-2 tensor $\tilde{\mathbf{T}}$, its *cofactor tensor* $\tilde{\mathbf{T}}^c$ can be obtained simply as the *dyad* of vector $\mathbf{e}_i \times \mathbf{e}_j$ and vector $(\tilde{\mathbf{T}}\cdot\mathbf{e}^i) \times (\tilde{\mathbf{T}}\cdot\mathbf{e}^i)$ such that

$$\tilde{\mathbf{T}}^c = [(\tilde{\mathbf{T}}\cdot\mathbf{e}^i) \times (\tilde{\mathbf{T}}\cdot\mathbf{e}^i)][\mathbf{e}_i \times \mathbf{e}_j] = \frac{1}{2}\epsilon_{skq}\epsilon^{mnp}T_n^{\ k}T_p^{\ q}\mathbf{e}^s\mathbf{e}_m. \tag{4.2.1}$$

Thus, the *1st invariant* $I_1(\tilde{\mathbf{T}})$ and *2nd invariant* $I_2(\tilde{\mathbf{T}})$ of tensor $\tilde{\mathbf{T}}$ can be given through the double dot product:

$$I_1(\tilde{\mathbf{T}}) = \tilde{I} \cdot\cdot \tilde{\mathbf{T}}, I_2(\tilde{\mathbf{T}}) = I_1(\tilde{\mathbf{T}}^c) = \tilde{I} \cdot\cdot \tilde{\mathbf{T}}^c = \frac{1}{2}[I_1^2(\tilde{\mathbf{T}}) - I_1(\tilde{\mathbf{T}}^2)]. \tag{4.2.2}$$

This process can be followed to, compute other invariants, obtain the Hamilton-Cayley relations, take tensor derivative of scalar and tensor functions and thus deduce constitutive relations from pertinent energy; essentially, using already known operations of dot and cross products on invariant objects, vectors inclusive, and many more operations.

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Hamilton-Kelly's Theorem

The set $\tilde{\mathbf{T}}^2, \tilde{\mathbf{T}}, \tilde{I}$, where $\tilde{\mathbf{T}}^0 \equiv \tilde{I}$, forms the basis for any tensor. $\tilde{\mathbf{T}}^m$ in the space $\tilde{\mathbf{T}} \in \mathsf{T}^{\,m} \subset \Omega(\mathsf{E}^n)$, $m \geq 3$ such that

$$\tilde{\mathbf{T}}^3 = I_1(\tilde{\mathbf{T}})\tilde{\mathbf{T}}^2 - I_2(\tilde{\mathbf{T}})\tilde{\mathbf{T}} + I_1(\tilde{\mathbf{T}})\tilde{I}. \tag{4.2.3}$$

i.e. the tensor $\tilde{\mathbf{T}}^m$, $m \geq 3$ is expressible linearly through tensors of lower degree $\tilde{\mathbf{T}}^2, \tilde{\mathbf{T}}, \tilde{I}$.

We further recall, in addition to (4.2.2), the following relations for nonsingular tensor $\tilde{\mathbf{T}}$

$$\tilde{\mathbf{T}}^{-1} = \frac{1}{I_1(\tilde{\mathbf{T}})}[\tilde{\mathbf{T}}^2 - I_1(\tilde{\mathbf{T}})\tilde{\mathbf{T}} + I_2(\tilde{\mathbf{T}})\tilde{I}], \quad I_3(\tilde{\mathbf{T}}) = \frac{1}{6}[I_1^3(\tilde{\mathbf{T}}) - 3I_1(\tilde{\mathbf{T}})I_1(\tilde{\mathbf{T}}^2) + 2I_1(\tilde{\mathbf{T}}^3)],$$

$$I_1(\tilde{\mathbf{T}}^{-1}) = \frac{I_2(\tilde{\mathbf{T}})}{I_3(\tilde{\mathbf{T}})},\ I_2(\tilde{\mathbf{T}}^{-1}) = \frac{I_1(\tilde{\mathbf{T}})}{I_3(\tilde{\mathbf{T}})},\ I_3(\tilde{\mathbf{T}}^{-1}) = \frac{1}{I_3(\tilde{\mathbf{T}})} = det(\tilde{\mathbf{T}}^{-1}).$$

We also note the following relations, including inequalities,

$$I_k(\tilde{\mathbf{T}}) = I_k(\tilde{\mathbf{T}}^t)\ k = 1,2,3; \quad I_1(\tilde{\mathbf{T}}) \geq 3I_3^{\frac{1}{3}}(\tilde{\mathbf{T}}),\ I_2(\tilde{\mathbf{T}}) \geq 3I_3^{\frac{2}{3}}(\tilde{\mathbf{T}}),\ I_1^2(\tilde{\mathbf{T}}) \geq 3I_2(\tilde{\mathbf{T}}). \tag{4.2.4}$$

Tensor Derivative of Invariants - Energy Function

Differentiation in Tensor Argument: Frechet Derivative}

Frechet Derivative

Here, the tensor derivative of an invariant is taken from the first principle of variation with respect to the {argument} in the sense of *Frechet*'s derivative.

Thus, we define derivative of a differentiable function $F \in C(\Omega)$ as the *linear coefficient of the variation of the argument due to the variation of the function*:

$$\delta F = F(x + \delta x) - F(h) = F'(x)\delta x, \tag{5.1.1}$$

where δF is the variation of the function $F(x)$ due to variation of the argument δx . Here, $F'(x)$ as the coefficient of δx in its linear form, constitutes the derivative of the function F . It is in this form that the notion of *Frechet Derivative* is invoked in respect of invariants/tensors [10].

Computation of Derivative of Scalars

Lemma: The derivative with respect to tensor $\tilde{\mathbf{T}}$ of the 1st, 2nd and 3rd invariants, $I_1(\tilde{\mathbf{T}})$, $I_2(\tilde{\mathbf{T}})$, $I_3(\tilde{\mathbf{T}})$, is given respectively as

$$I_1(\tilde{\mathbf{T}})_{\tilde{\mathbf{T}}} = \tilde{I},\ I_1(\tilde{\mathbf{T}}^2)_{\tilde{\mathbf{T}}} = 2\tilde{\mathbf{T}}^t,\ I_1(\tilde{\mathbf{T}}^3)_{\tilde{\mathbf{T}}} = 3\tilde{\mathbf{T}}^{2t}.$$

Proof

Derivative of Tensor Invariants.

a. We note that the first invariant of a tensor is a linear scalar function of its argument and it is given by the double dot product between the tensor and the unit tensor:

$$I_1(\tilde{\mathbf{T}}) \overset{(4.2.2)}{=} \tilde{I}\cdot\cdot\tilde{\mathbf{T}}.$$

Then, from the first principle we have

(i) $\delta I_1(\tilde{\mathbf{T}}) \overset{(5.1.1)}{=} I_1(\tilde{\mathbf{T}} + \delta\tilde{\mathbf{T}}) - I_1(\tilde{\mathbf{T}}) = \tilde{I}\cdot\cdot(\tilde{\mathbf{T}} + \delta\tilde{\mathbf{T}}) - \tilde{I}\cdot\cdot\tilde{\mathbf{T}} = \tilde{I}\cdot\cdot\delta\tilde{\mathbf{T}} = \tilde{I}\cdot\cdot\delta\tilde{\mathbf{T}}^t.\ \Rightarrow I_1(\tilde{\mathbf{T}})_{\tilde{\mathbf{T}}} = \tilde{I}.$

(ii) Likewise we have

$\delta I_1(\tilde{\mathbf{T}}^2) = I_1[(\tilde{\mathbf{T}} + \delta\tilde{\mathbf{T}})\cdot(\tilde{\mathbf{T}} + \delta\tilde{\mathbf{T}})] - I_1(\tilde{\mathbf{T}}^2) = I_1(\tilde{\mathbf{T}}\cdot\delta\tilde{\mathbf{T}} + \delta\tilde{\mathbf{T}}\cdot\tilde{\mathbf{T}})$,

and ignoring the nonlinear term $\delta\tilde{\mathbf{T}}\cdot\delta\tilde{\mathbf{T}}$ due definition,

$$= \tilde{I}\cdot\cdot\tilde{\mathbf{T}}\cdot\delta\tilde{\mathbf{T}} + \tilde{I}\cdot\cdot\delta\tilde{\mathbf{T}}\cdot\tilde{\mathbf{T}} = \tilde{\mathbf{T}}\cdot\cdot\delta\tilde{\mathbf{T}} + \tilde{\mathbf{T}}\cdot\cdot\delta\tilde{\mathbf{T}} = 2\tilde{\mathbf{T}}^{t}\cdot\cdot\delta\tilde{\mathbf{T}}^{t}. \;\Rightarrow\; I_1(\tilde{\mathbf{T}}^2)_{\tilde{\mathbf{T}}} = 2\tilde{\mathbf{T}}^{t}.$$

(iii) $\delta I_1(\tilde{\mathbf{T}}^3) = I_1(\tilde{\mathbf{T}}^2\cdot\delta\tilde{\mathbf{T}} + \tilde{\mathbf{T}}\cdot\delta\tilde{\mathbf{T}}\cdot\tilde{\mathbf{T}} + \delta\tilde{\mathbf{T}}\cdot\tilde{\mathbf{T}}^2) = 3\tilde{\mathbf{T}}^2\cdot\cdot\delta\tilde{\mathbf{T}} = 3\tilde{\mathbf{T}}^{2t}\cdot\cdot\delta\tilde{\mathbf{T}}^{t}. \Rightarrow I_1(\tilde{\mathbf{T}}^3)_{\tilde{\mathbf{T}}} = 3\tilde{\mathbf{T}}^{2t}.$

Hence,
$I_1(\tilde{\mathbf{T}})_{\tilde{\mathbf{T}}} = \tilde{I},\; I_1(\tilde{\mathbf{T}}^2)_{\tilde{\mathbf{T}}} = 2\tilde{\mathbf{T}}^{t}, I_1(\tilde{\mathbf{T}}^3)_{\tilde{\mathbf{T}}} = 3\tilde{\mathbf{T}}^{2t}.$

b. **Derivative of 2nd and 3rd Invariants**

By invoking the Hamilton-Kelly theorem relating tensor of any degree through degree not more than two *via-a-vis* the invariants of that tensor, we deduce the pertinent expressions for the derivative of higher invariants.

(iv) Indeed, in cognizance of (4.2.4) and the derivatives of first invariants indicated above, we obtain the required derivatives of the 2nd and 3rd invariants as

$$I_2(\tilde{\mathbf{T}})_{\tilde{\mathbf{T}}} = I_1(\tilde{\mathbf{T}})\tilde{I} - \tilde{\mathbf{T}}^{t}, \quad I_3(\tilde{\mathbf{T}})_{\tilde{\mathbf{T}}} = \tilde{\mathbf{T}}^{t2} - I_1(\tilde{\mathbf{T}})\tilde{\mathbf{T}}^{t} + I_2(\tilde{\mathbf{T}})\tilde{I} = I_3(\tilde{\mathbf{T}})(\tilde{\mathbf{T}}^{t})^{-1}.$$

Theorem: The derivative of any scalar function of tensor argument $\varphi(\tilde{\mathbf{T}})$ is

$$\varphi(I_1(\tilde{\mathbf{T}}), I_2(\tilde{\mathbf{T}}), I_3(\tilde{\mathbf{T}}))_{\tilde{\mathbf{T}}} = (\frac{\partial\varphi}{\partial I_1} + I_1(\tilde{\mathbf{T}})\frac{\partial\varphi}{\partial I_2} + I_2\frac{\partial\varphi}{\partial I_3})\tilde{I} - (\frac{\partial\varphi}{\partial I_2} + I_1\frac{\partial\varphi}{\partial I_3})\tilde{\mathbf{T}}^{t} + \frac{\partial\varphi}{\partial I_3}\tilde{\mathbf{T}}^{t2}.$$

Proof: Any scalar function of a tensor argument is ultimately expressed through the invariants of the tensor $\varphi(\tilde{\mathbf{T}}) = \varphi(I_1(\tilde{\mathbf{T}}), I_2(\tilde{\mathbf{T}}), I_3(\tilde{\mathbf{T}}))$. Invoking the last lemma, the Hamilton-Kelly theorem and the established derivatives of invariants, the theorem is proved:

$$\varphi(I_1(\tilde{\mathbf{T}}), I_2(\tilde{\mathbf{T}}), I_3(\tilde{\mathbf{T}}))_{\tilde{\mathbf{T}}} = [\frac{\partial\varphi}{\partial I_1} + I_1(\tilde{\mathbf{T}})\frac{\partial\varphi}{\partial I_2}]\tilde{I} - \frac{\partial\varphi}{\partial I_2}\tilde{\mathbf{T}}^{t} + \frac{\partial\varphi}{\partial I_3}I_3(\tilde{\mathbf{T}})(\tilde{\mathbf{T}}^{t})^{-1}$$

$$= (\frac{\partial\varphi}{\partial I_1} + I_1(\tilde{\mathbf{T}})\frac{\partial\varphi}{\partial I_2} + I_2\frac{\partial\varphi}{\partial I_3})\tilde{I} - (\frac{\partial\varphi}{\partial I_2} + I_1\frac{\partial\varphi}{\partial I_3})\tilde{\mathbf{T}}^{t} + \frac{\partial\varphi}{\partial I_3}\tilde{\mathbf{T}}^{t2}.$$

Theorem: Suppose the invariants of tensor, arising from geometry of deformation in elasticity, are

$$s_0 = \mathbf{c}\cdot(\tilde{U} - \tilde{I})\cdot\mathbf{c}, \quad s_4 = \mathbf{c}\cdot(\tilde{U} - \tilde{I})^2\cdot\mathbf{c},$$

where, $\tilde{U}$ such that $\tilde{U}^2 = \overset{o}{\nabla}\mathbf{R}\cdot\overset{o}{\nabla}\mathbf{R}^{t}$ is the symmetric *stretch tensor*, $\overset{o}{\nabla}\mathbf{R}$ is a non-symmetric rank-2 tensor of *deformation gradient* such that $\overset{o}{\nabla}\mathbf{R} = \tilde{U}\cdot\tilde{O}^{D}$, $\tilde{O}^{D} = \tilde{U}^{-1}\cdot\overset{o}{\nabla}\mathbf{R}$ is the *deformative rotation tensor*, $\tilde{I}$ is the unit tensor, $\mathbf{R}$ is the position vector and $\mathbf{c}$ is a unit directional vector. Then, the tensor derivative with respect to $\overset{o}{\nabla}\mathbf{R}$ of s_0 and s_4 are

$$s_{o_{\overset{o}{\nabla}\mathbf{R}}} = \frac{\partial s_o}{\partial\overset{o}{\nabla}\mathbf{R}} = \frac{1}{2}(\mathbf{cc}\cdot\tilde{O}^{D} + \tilde{U}^{-1}\cdot\mathbf{cc}\cdot)\overset{o}{\nabla}\mathbf{R} \quad \text{and}$$

$$s_{4_{\overset{o}{\nabla}\mathbf{R}}} = \frac{\partial s_4}{\partial\overset{o}{\nabla}\mathbf{R}} = 2\mathbf{cc}\cdot\overset{o}{\nabla}\mathbf{R} - \mathbf{cc}\cdot\tilde{O}^{D} - \tilde{U}^{-1}\cdot\mathbf{cc}\cdot\overset{o}{\nabla}\mathbf{R}.$$

Proof: The proof of this follows directly from the application of the previously enunciated procedure for Frechet derivative of a tensor, on any $\tilde{I}$, $\tilde{\mathbf{T}}$ and $\tilde{\mathbf{T}}^2$.

Theorem:

Suppose in addition to the invariants given in the previous theorem we have also

$$s_1 = I_1(\tilde{U} - \tilde{I}) = \tilde{I}\cdot\cdot(\tilde{U} - \tilde{I}) \quad \text{and} \quad s_2 = I_1(\tilde{U} - \tilde{I})^2 = \tilde{I}\cdot\cdot[(\tilde{U} - \tilde{I})\cdot(\tilde{U} - \tilde{I}).$$

Now let the scalar energy function W be given as

$$(i) W = \mu s_2 + \frac{1}{2}\lambda s_1^2, \quad (ii) W = \lambda s_2 + \frac{1}{2}\lambda_1 s_1^2 + \lambda_0 s_0,$$

respectively for an isotropic elastic material, for a plane transversely isotropic material [19-20].

Then the *Piola stress tensor* resulting from the derivation of W with respect to $\overset{o}{\nabla}\mathbf{R}$, is respectively

$$(i)\ \tilde{P} \equiv \frac{\partial W}{\partial \overset{o}{\nabla}\mathbf{R}} = 2\mu \overset{o}{\nabla}\mathbf{R} + [\lambda s_1 + 2\mu]\tilde{O}^D = [(\lambda s_1 - 2\mu)\tilde{U}^{-1} + 2\mu]\cdot\overset{o}{\nabla}\mathbf{R},$$

$$(ii)\ \tilde{P} \equiv \frac{\partial W}{\partial \overset{o}{\nabla}\mathbf{R}} = 2\lambda_2 \overset{o}{\nabla}\mathbf{R} + (\lambda_1 s_1 - 2\lambda_2)\tilde{O}^D + \lambda_0 \mathbf{cc}\cdot\overset{o}{\nabla}\mathbf{R},$$

where λ and μ are the so-called Lame's constants from elasticity, λ_0, λ_1 and λ_2 are pertinent constants associated with the anisotropy of the elastic composite materials [20].
Proof: The proof of this follows directly from the application of the procedure for Frechet derivative of a tensor on $\tilde{\mathbf{T}}$, $\tilde{\mathbf{T}}^2$ and $\tilde{I}$ and the previous theorem, straightforwardly.

Theory of Strain-Gradient-Divergence Plasticity
Plasticity is an aspect of elasticity, when irreversible process takes place. Here, the robustness of tensor operations and the concepts therein, such as: scalar functions and invariants and their tensor derivatives; multiple scalar products and conjugacy between geometric and mechanical characteristics have been deftly applied to upgrade the theory of strain-gradient plasticity of Gurtin and Anand [17] to obtain the theory of strain-gradient-divergence plasticity [16].

Now, consider a body Ω undergoing plastic deformation. Suppose $\mathbf{u}(\mathbf{x},t)$ denotes the displacement vector of an arbitrary point $\mathbf{x}$ in a region Ω describing a body composed of manifolds of particles. The classical theory of isotropic plastic solids undergoing small deformations is based on the kinematic relations given by the decomposition of the displacement gradient;

$$\nabla\mathbf{u} = \tilde{H}^e + \tilde{H}^p; \quad tr\tilde{H}^p = I_1(\tilde{H}^p) = \tilde{H}^p\cdot\cdot\tilde{I} = 0. \tag{6.1}$$

$$\tilde{E}^e = \frac{1}{2}(\tilde{H}^e + \tilde{H}^{et}); \ \tilde{E}^p = \frac{1}{2}(\tilde{H}^p + \tilde{H}^{pt}), \tag{6.2}$$

where $\tilde{E}^e$ represents rotation and stretching while $\tilde{E}^p$ denotes the plastic distortion characterizing the evolution of dislocations and other defects through the structure. $I_1(\tilde{H}^p) = 0$ defines the condition of plastic incompressibility. The elastic and plastic strains are defined by (6.2), while $\tilde{W}^e = skw\tilde{H}^e$ and $\tilde{W}^p = skw\tilde{H}^p$ are the elastic and plastic rotation tensor respectively.
Let the internal and external virtual power expenditure over a micro-region $P \subset \Omega$ be

$$W_{int} = \int_P [\tilde{\mathbf{T}}\cdot\cdot\tilde{E}^e + \boldsymbol{\chi}\cdot\nabla\cdot\tilde{\mathbf{E}}^{\mathbf{p}} + \tilde{\mathbf{T}}^{\mathbf{p}}\cdot\cdot\tilde{\mathbf{E}}^{\mathbf{p}} + \tilde{\mathbf{K}}^{\mathbf{p}} \,\vdots\, \nabla\tilde{\mathbf{E}}^{\mathbf{p}}]dV;$$

$$W_{ext}(P,V) = \int_P [\mathbf{t}(\mathbf{n})\cdot\mathbf{u} + \mathbf{K}(\mathbf{n})\cdot\cdot\tilde{E}^p]dA + \int_{\partial P}\mathbf{b}\cdot\mathbf{u}dV,$$

for which, given the set of virtual velocities $\vartheta = (\delta\dot{\mathbf{u}}, \delta\dot{\tilde{H}}^e, \delta\dot{\tilde{E}}^e)$, by the principle of virtual power we have $W_{int}(P,\vartheta) = W_{ext}(P,\vartheta)$, which results in the set of *macrobalance force* with the corresponding *macrotraction* and *microbalance force* with the corresponding *microtraction*:

$$\nabla\cdot\tilde{\mathbf{T}} + \mathbf{b} \ \text{ and } \ \tilde{\mathbf{T}}\cdot\mathbf{n} = \mathbf{t}, \tag{6.3}$$

$$\tilde{\mathbf{T}}_o = \tilde{\mathbf{T}}^p - sym_o(\nabla\boldsymbol{\chi}) - \nabla\cdot\tilde{K}^p \ \text{ and } \ \tilde{K} = sym_o(\boldsymbol{\chi}\otimes\mathbf{n}) + \tilde{K}^p\cdot\mathbf{n}, \tag{6.3'}$$

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where $\tilde{\mathbf{T}}_o = \tilde{\mathbf{T}} - \frac{1}{3}(\mathrm{tr}\tilde{\mathbf{T}})\tilde{I}$ is the deviatoric part of the macrostress $\tilde{\mathbf{T}}$, $sym_o(\nabla\boldsymbol{\chi})$ is the symmetric deviatoric part of the gradient of the microforce $\boldsymbol{\chi}$, while $\boldsymbol{\chi}\otimes\mathbf{n}$ is a dyad. Note that symmetric-deviatoric part of tensor $\tilde{\mathbf{T}}$ is $sym_o\tilde{\mathbf{T}} \equiv [\frac{1}{2}(\tilde{\mathbf{T}} + \tilde{\mathbf{T}}^t) - \frac{1}{3}(\mathrm{tr}\tilde{\mathbf{T}})\tilde{I}]$.

Here, the internal microforce $\boldsymbol{\chi}$ is energy conjugate to $\nabla\cdot\tilde{E}^p$, macrostress $\tilde{\mathbf{T}}$ is energy conjugate to $\tilde{E}^e$, microstress $\tilde{\mathbf{T}}^p$ is energy conjugate to $\tilde{E}^p$, the polar microstress rank3 tensor $\tilde{K}^p$ is energy conjugate to the rate of plastic strain gradient $\nabla\tilde{E}^p$. $\mathbf{b}$ is a body force in P, $\mathbf{t}$ is a microtraction vector and $\mathbf{K}$ a rank2 microtraction tensor both on ∂P, $\mathbf{n}$ is the outward unit normal vector, orientating the surface ∂P.

By the second law of thermodynamics, the free energy imbalance for this plastic process is taken, due (6.3), as

$$\psi \leq \tilde{\mathbf{T}}\cdot\cdot\tilde{E}^e + \boldsymbol{\chi}\nabla\cdot\tilde{E}^p + \tilde{\mathbf{T}}^p\cdot\cdot\tilde{E}^p + \tilde{K}^p \vdots \nabla\tilde{E}^p,$$

which nudges us to take \$\psi\$ as the Helmholtz free energy in the form

$$\psi = \psi(\tilde{E}^e, \tilde{E}^p, \nabla\cdot\tilde{E}^p, \nabla\times\tilde{E}^p). \tag{6.4}$$

Following from the previous theory, we then take the tensor derivative of the scalar (i.e. the free energy) with respect to each argument to obtain the corresponding conjugate mechanical parameter:

$$\tilde{\mathbf{T}} = \frac{\partial\psi}{\partial\tilde{E}^e}, \quad \tilde{\mathbf{T}}^p = \frac{\partial\psi}{\partial\tilde{E}^p} \quad \boldsymbol{\chi}_{en} = \frac{\partial\psi}{\partial\nabla\cdot\tilde{E}^p}, \quad (\tilde{K}_{en}{}^p)_{jqp} = (sym_o P)_{jqp}, \tag{6.5}$$

where $(P)_{jqp} = \frac{\partial\psi}{\partial G_{ij}}\epsilon_{ipq}$ and $G_{ij} = \epsilon_{irs}E^p_{js,r}$.

Consequently, the energy is obtained in the quadratic form as

$$\frac{1}{2}\kappa I_1^2(\tilde{E}^e) + \mu|\tilde{E}^e_o|^2 + \frac{1}{2}\mu|\tilde{E}^p|^2 + \frac{1}{2}\mu Q^2|\nabla\cdot\tilde{E}^p|^2 + \frac{1}{2}\mu L^2|\nabla\times\tilde{E}^p|^2, \tag{6.6}$$

where Q and L are called energetic length scales associated with $\nabla\cdot\tilde{E}^p$ and $\nabla\times\tilde{E}^p$ respectively. The dissipative microstresses based on the von Mises yield criterion are given in terms of their corresponding power conjugates as follows

$$\boldsymbol{\chi}_{\mathrm{dis}} = q^2 S_o(\frac{d^p}{d_o})^m \frac{\nabla\cdot\tilde{E}^p}{d^p}; \tilde{\mathbf{T}}^p_{\mathrm{dis}} = S_o(\frac{d^p}{d_o})^m \frac{\tilde{E}^p}{d^p}; \tilde{K}^p_{\mathrm{en}} = l^2 S_o(\frac{d^p}{d_o})^m \frac{\nabla\tilde{E}^p}{d^p}, \tag{6.7}$$

S_o is the initial yield strength, d_o is the initial flow rate, m is the rate sensitivity parameter and d^p is the effective flow rate defined by

$$d^p = \sqrt{|\tilde{E}^p|^2 + q^2|\nabla\cdot\tilde{E}^p|^2 + q^2|\nabla\tilde{E}^p|^2},$$

where q and l are called dissipative length scales associated with $\nabla\cdot\tilde{E}^p$ and $\nabla\tilde{E}^p$ respectively. Putting (6.5) and (6.7) into the microforce balance and microtraction condition $(6.3)'$, we obtain the interested viscoplasticity flow rule, which is a generalization of the Gurtin-Anand model:

$$\tilde{\mathbf{T}}_o + \mu(L^2\Delta\tilde{E}^p + (Q^2 - L^2)sym_o(\nabla\nabla\tilde{E}^p) - \tilde{E}^p) = S_o(\frac{d^p}{d_o})^m\frac{\tilde{E}^p}{d^p} - q^2 S_o(sym_o(\nabla[(\frac{d^p}{d_o})^m\frac{\nabla\cdot\tilde{E}^p}{d^p}])) - l^2 S_o\nabla[(\frac{d^p}{d_o})^m\frac{\nabla\tilde{E}^p}{d^p}].$$

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