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A single-variable approach for layered beams with imperfect interfaces

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Abstract. The first-order shear deformable laminated beam problem is reformulated in terms of a fictitious bending displacement as primal variable. A fourth-order differential equation governs the problem with two additional unknown constants coming from the integration of the longitudinal displacement. With the classical six boundary conditions, the problem is complete and well posed. An isogeometric collocation scheme is developed to solve the problem numerically. The formulation is completely locking-free and satisfies high continuity requirements for the approximation functions. The results for an exemplary structure confirm the validity and the good performance of the method, which is preliminary to the single variable reformulation of a more accurate zig-zag model for laminates with perfect and imperfect interfaces.

Introduction

As is well known, the elastic and thermal mismatch between the layers of composite beams induces complex zig-zag distributions of stresses and displacements in the thickness direction; relative displacements may also arise at the layer interfaces. Alternative approaches to the classical structural theories, which are based on assumptions that make them inadequate, must be used.

The zig-zag theories overcome the drawbacks of equivalent single-layer and layer-wise theories. A global displacement field is enriched by through-the-thickness zigzag functions. This allows to take into account the local effects due to material inhomogenities and to impose stress continuity at the interfaces, while maintaining the number of degrees of freedom low. Originally formulated zigzag theories had some inconsistencies and limitations and were not ideal for the implementation in numerical solution schemes, leading to the development of refined theories, which however also have some important drawbacks (see e.g. [1] for a review).

The validity of a more recent, energetically consistent, multiscale model formulated in [2] was demonstrated for a wide class of problems: plane-strain problems [3]; beams and plates subjected to thermal loadings [4]; propagation of plane-strain harmonic waves [5]; brittle delamination fracture under mode II dominant conditions [6-8].

Our work proposes a novel single-variable approach to the zigzag model in [4], by extending to laminates with and without imperfect interfaces a technique recently developed for homogeneous beams [9,10], as an efficient alternative approach to refined zigzag theories. All details and results will be presented at the conference; here, we describe the methodology and show some preliminary promising results in the framework of classical first-order shear deformation theory, which confirm the validity and efficacy of the method.

Equations governing first-order shear deformation theory for laminates

In this section the equations governing first-order shear deformation theory for laminates are briefly recalled with reference to the multi-layered beam shown in Fig. 1 and having thickness h, unit width in the direction x_1 and length L in the direction x_2. In the system of Cartesian coordinates x_1, x_2, x_3, the plane x_3=0 defines the reference surface of the beam, which is arbitrarily chosen. The

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beam consists of n layers joined by $n-1$ perfectly bonded interfaces (no relative sliding and transverse displacements are allowed). Each layer is linearly elastic and orthotropic with principal material axes parallel to the geometrical axes; the layers can have different mechanical properties. Coordinate x_3^k defines the interface k, that is the upper surface of the layer k having thickness ${}^{(k)}h$ (with k=1,..,n numbered from bottom to top).

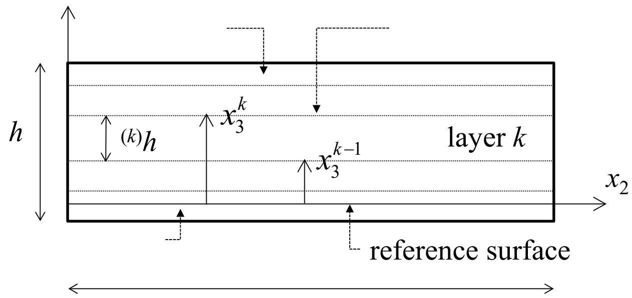

Fig. 1 *Multi-layered beam geometry*

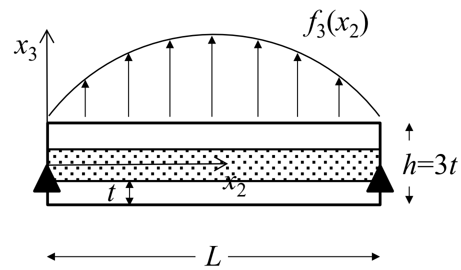

Fig. 2 *Analyzed exemplary beam*

The beam is subjected to boundary conditions and mechanical loads acting on the upper and lower surfaces and on the lateral bounding surface, and applied so as to satisfy plane strain conditions in the plane x_2-x_3 perpendicular to the axis x_1.

Transverse normal stresses are set equal to zero ($\sigma_{33}=0$) and the in plane displacements are assumed as linear functions of the through-the-thickness coordinate. Then, the global equilibrium equations can be written in terms of the global degrees of freedom v_{02}, w_0 and φ_2 (denoting, respectively, the displacements of the reference axis, along the directions x_2 and x_3, and the cross section rotation, about the axis x_1). Using a comma followed by subscripts to denote derivatives with respect to the corresponding coordinates, they are:

$$\begin{aligned}
&A_{22}v_{02,22}+B_{22}\varphi_{2,22}+f_2=0\\
&B_{22}v_{02,22}+D_{22}\varphi_{2,22}-k_{44}C_{44}^{P}(\varphi_2+w_{0,2})+f_{2m}=0\ ,\\
&-k_{44}C_{44}^{P}(\varphi_{2,2}+w_{0,22})-f_3=0
\end{aligned}\tag{1}$$

with coefficients

$$(A_{22},B_{22},D_{22})=\sum\nolimits_{k=1}^{n}\int_{x_3^{k-1}}^{x_3^k}{}^{(k)}\overline{C}_{22}(1,x_3,x_3^2)\mathrm{d}x_3 \quad\text{and}\quad C_{44}^{P}=\sum\nolimits_{k=1}^{n}\int_{x_3^{k-1}}^{x_3^k}{}^{(k)}C_{44}\mathrm{d}x_3\ ,\tag{2}$$

where: for the layer k, ${}^{(k)}\overline{C}_{22}={}^{(k)}(C_{22}-C_{23}C_{32}/C_{33})$ relates local longitudinal normal stress and strain, through ${}^{(k)}\sigma_{22}={}^{(k)}\overline{C}_{22}\,{}^{(k)}\varepsilon_{22}$ under the assumed plane-strain conditions (${}^{(k)}C_{ij}$ with i,j=1,2 being the coefficients of the 6x6 stiffness matrix), and ${}^{(k)}C_{44}$ is the stiffness coefficient relating local transverse shear stress and strain, through ${}^{(k)}\sigma_{23}={}^{(k)}C_{44}2\,{}^{(k)}\varepsilon_{23}$; f_2, f_3, f_{2m} denote distributed longitudinal and transverse global loads (positive if rightward and upward, respectively) and global couples (positive if clockwise) acting on the reference surface. In Eq. (1) a shear correction factor k_{44} is introduced to improve the treatment of shear, given the limitations of the assumed theory to describe the global kinematics. The coefficients A_{22} and D_{22} are the extensional and bending laminate stiffnesses, B_{22} is the coupling stiffness of the laminate and relates bending strain with normal force and vice-versa.

Eq. (1) is a system of coupled differential equations in three unknowns, which is completed by six boundary conditions imposed at the ends x_2=0, L with outward normal $\mathbf{n}=\{0,\mp 1,0\}^{\mathrm{T}}$:

$$
\begin{aligned}
&v_{02}=\tilde{v}_{02} \quad \text{or} \quad N_{22}n_2=[A_{22}v_{02,2}+B_{22}\varphi_{2,2}+w_{0,22}]n_2=\tilde{N}_{22}n_2\\
&w_0=\tilde{w}_0 \quad \text{or} \quad Q_{22}n_2=k_{44}C_{44}^P(\varphi_2+w_{0,2})n_2=\tilde{Q}_{22}n_2\\
&\varphi_2=\tilde{\varphi}_2 \quad \text{or} \quad M_{22}n_2=[B_{22}v_{02,2}+D_{22}\varphi_{2,2}]n_2=\tilde{M}_{22}n_2
\end{aligned}
\tag{3}
$$

(the tilde defines prescribed values). Once the problem has been solved, the local response for each layer is obtained straightforwardly through local compatibility, constitutive and equilibrium equations.

Single-variable formulation

In this section a single-variable formulation of the problem under consideration is presented, where the system of coupled differential equations (1) is reduced to a single equation in one unknown variable. As discussed in [9,10], different choices are possible for the primal variable. The approach proposed in [10] is followed here and the global transverse displacement w_0 is splitted into two parts: a bending part, w_{0b}, and a shear part, w_{0s}, which are defined as follows

$$
w_0=w_{0b}+w_{0s} \quad \text{with} \quad \varphi_2=-w_{0b,2} \quad \text{so that} \quad \varphi_2+w_{0,2}=w_{0s,2}\,. \tag{4}
$$

Substituting Eq. (4) into the first and second of Eq. (1) yields $v_{02,22}$ and $w_{0s,2}$ as functions of $w_{0b,222}$, which can be derived once with respect to x_2 and introduced into the third of Eq. (1) to obtain a fourth-order differential equation in the one unknown w_{0b}

$$
\left[D_{22}-\left(B_{22}\right)^2(A_{22})^{-1}\right]w_{0b,2222}=f_3-B_{22}(A_{22})^{-1}f_{2,2}+f_{2m,2}\,. \tag{5}
$$

When w_{0b} is determined, the global rotation φ_2 follows directly from the second of Eq. (4), whereas the global transverse and longitudinal displacements w_0 and v_{02} are given by , respectively,

$$
\begin{aligned}
&w_0=w_{0b}-(k_{44}C_{44}^P)^{-1}\left\{\left[D_{22}-\left(B_{22}\right)^2(A_{22})^{-1}\right]w_{0b,22}+\int[B_{22}(A_{22})^{-1}f_2-f_{2m}]\mathrm{d}x_2\right\}\\
&v_{02}=B_{22}(A_{22})^{-1}w_{0b,2}-\iint f_2(A_{22})^{-1}\mathrm{d}x_2\mathrm{d}x_2+c_5x_2+c_6
\end{aligned}
, \tag{6}
$$

where the integration of v_{02} requires the introduction of two additional arbitrary constants, say c_5 and c_6. To complete the boundary value problem, substituting these results into Eq. (3) leads to the following expressions for the resultant forces and moment in terms of w_{0b}

$$
\begin{aligned}
&N_{22}n_2=[-\int f_2\mathrm{d}x_2+A_{22}c_5]n_2\\
&Q_{22}n_2=\left\{-[D_{22}-(B_{22})^2(A_{22})^{-1}]w_{0b,222}-B_{22}(A_{22})^{-1}f_2+f_{2m}\right\}n_2\\
&M_{22}n_2=\left\{-[D_{22}-(B_{22})^2(A_{22})^{-1}]w_{0b,22}-B_{22}(A_{22})^{-1}\int f_2\mathrm{d}x_2+B_{22}c_5\right\}n_2
\end{aligned}
\,. \tag{7}
$$

Eq.s (5)-(7) , together with the second of Eq. (4), represent a single-variable formulation for laminates. All kinematic and static variables are expressed in terms of only one unknown and these expressions do not contain integrals of the unknown funtion. This excludes a priori locking problems and facilitates the employment in discrete numerical solution schemes. Furthermore, analogously to Bernoulli–Euler beam theory, this formulation is rotation-free, but shear deformability is accounted for. For symmetric laminates, when a mid-thickness reference surface x_3=0 is chosen, B_{22}=0 and Eq.s (5)-(7) reduce to those of the single-variable formulation for homogeneous Timoshenko beams in [10], but with f_2 and $f_{2m}\neq 0$. In this case the membrane and bending equilibrium problems are decoupled and the global longitudinal displacement v_{02} is independent of the bending transverse displacement w_{0b}.

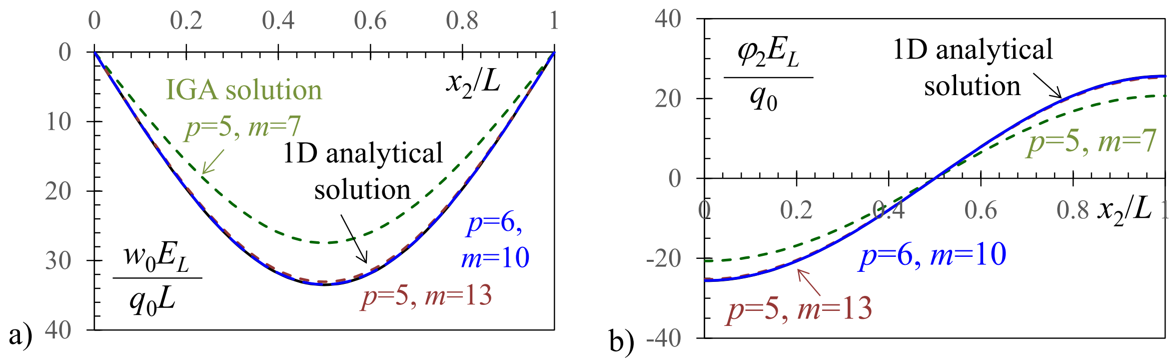

Fig. 3 Global response of the laminate: a) transverse displacement; b) rotation

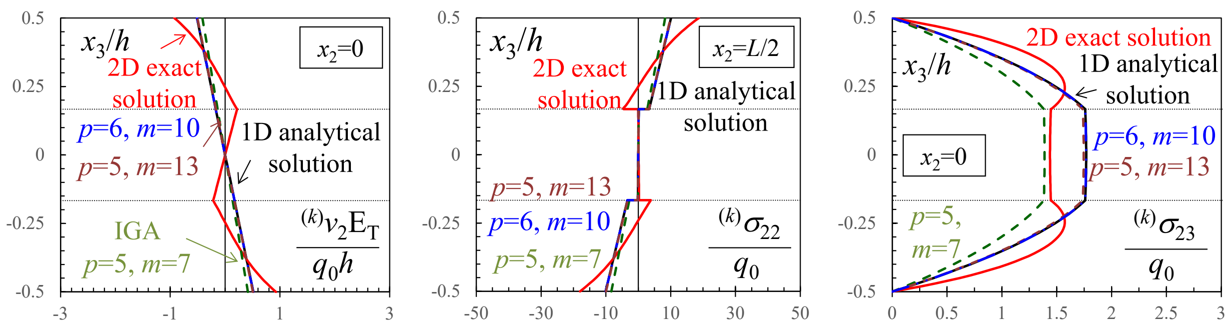

Fig. 4 Local response of the laminate: longitudinal displacement and in-plane stresses

Numerical solution through an isogeometric collocation scheme

In this section we develop an isogeometric collocation scheme (see, e.g., [10] and [11]) to solve numerically the laminate problem governed by the single-variable formulation derived above. In order to do this, the unknown primal variable w_{0b} is approximated by a B-spline curve, that is a linear combination of m piecewise polynomial basis functions of degree p, say $N_{i,p}$ with i=1,..,m, generated from a so-called knot vector and defined recursively starting with piecewise constants:

$$w_{0b} \approx \sum_{i=1}^{m} N_{i,p}(\xi = x_2 / L)\hat{w}_{0b_i} = \underline{N}\underline{\hat{w}}_{0b}$$
$$\text{so that} \quad \frac{\mathrm{d}^k w_{0b}}{\mathrm{d}x_2^k} \approx \sum_{i=1}^{m} \frac{\mathrm{d}^k N_{i,p}(\xi)}{\mathrm{d}\xi^k} L^{-k}\hat{w}_{0b_i} = L^{-k}\underline{N}^{(k)}\underline{\hat{w}}_{0b}, \tag{8}$$

where $\underline{N}$ is a row matrix of the B-spline shape functions and $\underline{\hat{w}}_{0b}$ is the vector of control variables which represent the m degreees of freedom of the numerical model (the *mesh*). A knot vector is a set of m+p+1 nondecreasing real numbers (the *knots*); for 1D straight geometries, they represent dimensionless coordinates in the direction x_2, say ξ=x_2/L, and divide the laminate domain (a *patch*) in m+p subintervals (the *knot spans*). The generation of the basis functions starts from a set of m+p piecewise constant functions, which =1 in a single knot span and =0 otherwise, and requires the employment of the recursive formula p times. In order to have interpolatory basis functions at knots located at the ends of the patch, where boundary conditions are imposed, open knot vectors with the first and last knot appearing p+1 times are considered.

Using Eq. (8) into Eq.s (5)-(7) gives global kinematic and static variables in a discrete form, whereas N_{22} is always given by the first of Eq. (7), since it is independent of w_{0b}:

$$
\begin{aligned}
&w_0 \approx \left\{ \underline{N} - (k_{44}C_{44}^P)^{-1}\left[D_{22} - (B_{22})^2 (A_{22})^{-1} \right] L^{-2}\underline{N}'' \right\} \underline{\hat{w}}_{0b} - (k_{44}C_{44}^P)^{-1}\int [B_{22}(A_{22})^{-1} f_2 - f_{2m}]\mathrm{d}x_2, \\
&\varphi_2 \approx -L^{-1}\underline{N}'\underline{\hat{w}}_{0b}, \\
&v_{02} \approx B_{22}(A_{22})^{-1}L^{-1}\underline{N}'\underline{\hat{w}}_{0b} - \iint f_2 (A_{22})^{-1}\mathrm{d}x_2\mathrm{d}x_2 + c_5x_2 + c_6, \\
&Q_{22}n_2 \approx \left\{ -[D_{22} - (B_{22})^2(A_{22})^{-1}]L^{-3}\underline{N}'''\underline{\hat{w}}_{0b} - B_{22}(A_{22})^{-1} f_2 + f_{2m} \right\} n_2, \\
&M_{22}n_2 \approx \left\{ -[D_{22} - (B_{22})^2(A_{22})^{-1}]L^{-2}\underline{N}''\underline{\hat{w}}_{0b} - B_{22}(A_{22})^{-1}\int f_2\mathrm{d}x_2 + B_{22}c_5 \right\} n_2.
\end{aligned}
\tag{9}
$$

To solve the problem, the strong form in Eq. (5) is discretized and evaluated at a set of suitable collocation points, say $\overline{\xi}_i$,

$$
\left[D_{22} - (B_{22})^2 (A_{22})^{-1} \right] L^{-4}\underline{N}''''(\overline{\xi}_i)\underline{\hat{w}}_{0b} = f_3(\overline{\xi}_i) - B_{22}(A_{22})^{-1} f_{2,2}(\overline{\xi}_i) + f_{2m,2}(\overline{\xi}_i). \tag{10}
$$

Following [11], the so-called Greville abscissae related to the fourth derivative space, which corresponds to the order of the differential equation, are here chosen; they are defined as follows

$$
\overline{\xi}_i = \frac{\xi_{i+5} + \xi_{i+6} + .. + \xi_{i+p}}{p-4} \quad \text{for } i = 1,..,m-4. \tag{11}
$$

Eq. (10) gives m−4 equations in the m unknown control variables. The missing equations are obtained by imposing 6 boundary conditions at the ends of the patch ($\overline{\xi}_i = 0,1$) through Eq. (9), which contain the two additional constants c_5 and c_6. To conclude, the discrete problem is governed by a system of m+2 algebric equations in the m+2 unknowns $\underline{\hat{w}}_{0b}$ and c_5, c_6.

Application to an exemplary case and conclusive remarks

The single-variable formulation and related numerical scheme is employed for the solution of the classical problem shown in Fig. 2 and solved in closed form in [12]. A cross-ply laminated beam is simply supported at the ends and subjected to a sinusoidal transverse load $f_3(x_2) = q_0 \sin(\pi x_2 / L)$ (f_2=f_{2m}=0). The beam consists of n=3 unidirectionally reinforced laminae symmetrically arranged with stacking sequence (0°/90°/0°) and perfectly bonded together. Each lamina has thickness t (h=3t) and is orthotropic with elastic constants E_L, E_T=E_L/25, G_{LT}=E_L/50, G_{TT}=E_L/125, ν_{LT}=ν_{TT}=0.25. A mid-thickness reference surface is chosen.

For the discretization, quintic (p=5) and higher degree (p=6) B-splines are used, since the governing differential equation is of fourth order. Three meshes with different number of degrees of freedom (m=7,13 for p=5 and m=10 for p=6) are considered.

The dimensionless global transverse displacement and rotation, as functions of the dimensionless position along the axis x_2, and local longitudinal displacement, and bending and transverse shear stresses, through the thickness, obtained for k_{44}=1 and L=4h are shown in Fig.s 3 and 4. The numerical results for discretizations p=5 with m=7 (green curves) and m=13 (brown curves), and p=6 with m=10 (blue curves) are presented together with the 1D analytical solution of Eq. (5) (black curves). The analytical and numerical results are in very good agreement, but for the coarsest discretization with p=5 and m= 7 in Fig. 3 and for transverse shear stresses in Fig. 4. In the latter figure the comparison with the 2D exact solution (red curves) allows to conclude that the solution for the highest-order and finest discretizations (p=5, m=13 and p=6, m=10) are statically equivalent, even though locally different, to the 2D exact solution; this is not true for p=5, m= 7. Extension of the technique to the zigzag model in [4], which is in progress, is expected to provide full agreement with the 2D solution.

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