Materials Research Proceedings **26** (2023) 405-410 https://doi.org/10.21741/9781644902431-66

Using multiple singular values in topology optimization of dynamic systems

Paolo Venini

Department of Civil Engineering and Architecture, Via Ferrata 3, 27100 Pavia, ITALY

paolo.venini@unipv.it

Keywords: Topology Optimization, Dynamic Systems, Matrix Norms, SVD

Abstract. Within the general framework of frequency-domain topology optimization of Multi Input-Multi Output (MI-MO) dynamic systems, suitable norms of the input/output transfer matrix are introduced as possible merit functions to be minimized. Among them, the by now classical H_∞-norm (i.e. the supremum of the maximum singular value over the whole frequency range), and the so-called *nuclear norm* (i.e. the sum of all the positive singular values are considered. Heuristic motivations are given that suggest which norm should one choose according to the practical objective to be pursued alongside a few numerical examples on topology optimization of 2D linear-elastic multiload SI-SO and MI-MO dynamic systems.

Introduction and Motivation

Topology optimization of dynamical systems presents a few peculiarities among which the most important ones may be listed as follows:

- Time versus frequency domain approaches. This preliminary choice gives rise to two different families of solving approaches that do not share much but, possibly but not certainly, the final result. As a matter of fact, time domain approaches lead to optimal designs that depend strongly on the specific choice of the time-dependent loading functions [1]. On the other side, frequency domain methods work on the frequency response function that is expected to enjoy a few desirable features, at least in the frequency range of interest.
- Focusing the attention on frequency-domain approaches, [2] minimizes the dynamic compliance of the system via an incremental frequency approach that operates at low or high value of the excitation frequency whereas [3] sets the problem as a minimum H_∞-norm of the frequency response function, in a sense broadening to open-loop systems the well-established H_∞-norm-based active control strategy in a closed-loop feedback framework.

From a practical point of view, (for SI-SO systems) the H_∞-norm of the frequency response function is the peak of the function itself and is therefore crystal clear the motivation behind the adoption of such a design approach. However, looking at the methodology from a more algebraic perspective is likely to shed new light onto the method itself and opens the way to a few potentially useful extensions that are in fact the object of this contribution. The idea is then to define a novel goal function that depends on a few singular values (and not only on the first one as is the case when the minimization of the H_∞-norm is pursued). *Mutatis mutandis*, there are similarities with those eigenvalue optimization strategies that, at least to overcome the singularity of the min-max eigenvalue problem, introduce a goal function that depends on a few eigenvalues, see e.g. [4]. By so doing, eigenvalue crossing in the design space is no longer an issue, loss of regularity does not show up and standard gradient-based approaches are shown to work properly.

Having in mind a MI-MO rectangular frequency-response matrix, the usefulness of its Singular Value Decomposition (SVD) for a full understanding of the dynamic features of the system is highlighted next. The physical meaning of the singular values and associated left and right singular vectors is described along with the algebraic relation between singular values and H_∞-norm.

Materials Research Proceedings **26** (2023) 405-410 https://doi.org/10.21741/9781644902431-66

Reference is made to [5, 6] for a comprehensive exposition of such concepts from an algebraic as well computational point of view.

The Singular Value Decomposition

To start off, a formal definition of SVD of a rectangular, possibly non-symmetric, matrix is given next.

Singular Value Decomposition. Let G be an m by n (possibly complex valued) matrix. Two sets of singular vectors exist such that:

- n right singular vectors $v_1, \dots, v_n$ are orthogonal to each other in $\mathbb{R}^n$;
- m left singular vectors $u_1, \dots, u_m$ are orthogonal to each other in $\mathbb{R}^m$;
- left and right singular vectors are connected by an "eigen-like" relation $Av = \sigma u$ that may be written component wise as

$$Gv_1 = \sigma u_1, \dots, Gv_r = \sigma_r; \quad Gv_{r+1} = 0, \dots, Gv_n = 0, \tag{1}$$

  where r is the *rank* of G and one may show that there exist r non-negative singular values that are usually cast in descending order: $\sigma_1 \leq \sigma_2 \leq \cdots \leq \sigma_r$.

The singular value decomposition of G finally reads:

$$A = U\Sigma V^T. \tag{2}$$

More explicitly, the matrix G may be written as a finite sum of rank-1 matrices as:

$$G = U\Sigma V^T = \sigma_1 u_1 v_1^T + \cdots + \sigma_r u_r v_r^T. \tag{3}$$

The following two propositions show a few reasons why the first singular values (and not only the largest one) are worth being investigated.

Theorem of Eckart-Young. Let $G_k = \sigma_1 u_1 v_1^T + \cdots \sigma_k u_k v_k^T$ be the *rank*-k SVD approximant to G. Then G_k is the overall best *rank*-k approximant to G, i.e.

$$||G - G_k|| \leq ||G - B|| \ \ \forall B \text{ with rank } k. \tag{4}$$

On the properties and computation of the second-largest and further singular values. The solution of the Problem:

$$\max_x \frac{||Gx||}{||x||} \text{ under the condition } v_1^T x = 0, \tag{5}$$

is σ_2 at $x = v_2$.

Matrix Norms, Singular Values and H_∞-Norm of a Transfer Function Matrix

Let p be the design variable vector (that contains the element densities in a topology optimization framework) and $G(i\omega, p)$ the frequency response matrix function. For the sake of introducing a few key concepts in algebra of non-square and non-symmetric matrices, it is convenient to fix both the frequency ω and the design variable vector p, say $\omega = \omega^*$ and $p = p^*$, so that $G^* = G(i\omega^*, p^*)$ is any complex-valued rectangular matrix.

The 2-norm of a matrix G^*. The *"largest growth factor"* concept [5] appears to be the most natural to introduce the norm of a transfer function matrix that governs the input/output relation of

Materials Research Proceedings **26** (2023) 405-410 https://doi.org/10.21741/9781644902431-66

a dynamical systems. In a more analytic format we may also refer to matrix norms induced by underlying (and previously defined) vector norms. One may write

$$||G^*|| = \sup_{v \neq 0} \frac{||G^* v||}{||v||}, \tag{6}$$

where each vector norm at the right hand side induces a corresponding matrix norm at the left hand side. If the 2-norm is used, one may show that

$$||G^*||_2 = \sigma_1, \tag{7}$$

i.e. the 2-norm of a matrix is its largest singular value. If now the dependence of G^* on the frequency ω is recovered, one may quickly realize that the H_∞-norm of a frequency-response matrix (that from an engineering point of view is the maximum amplification factor over the whole frequency axis, i.e. the *"largest growth factor"*) is defined as the supremum of the first singular value of $G(i\omega, p^*)$ with respect to the whole frequency axis, i.e.

$$||G(i\omega, p^*)||_\infty = \sup_{\omega \in (0,\infty)} \sigma_1(\omega). \tag{8}$$

It is interesting to note that the H_∞-norm of a matrix transfer function is in fact a 2-norm maximized over the frequency axis. From a numerical point of view, computing the H_∞-norm is quite a hard task as is the evaluation of the frequency at which the norm itself is attained. Alongside the H_∞-norm that depends exclusively on the largest (first) singular value, two more norms are theoretically suitable as goal functions when optimizing the topology of dynamical systems, i.e. the Frobenius norm and the Nuclear (the one adopted herein) norm that are respectively defined as:

$$||G||_F = \sqrt{\sigma_1^2 + \sigma_2^2 + \cdots + \sigma_r^2}, \quad ||G||_N = \sigma_1 + \sigma_2 + \cdots + \sigma_r. \tag{9}$$

Numerical Studies

Input data. As for the geometry of the problems investigated herein reference is made to Fig. 1 that also shows loads and constraints. By now standard aspects of topology optimization such as the SIMP idealization to handle intermediate materials, the nonlocal filters that are adopted to avoid checkerboarding, the finite elements that are used to derive the discrete version of the problem, not to mention the numerical scheme that is used to solve the optimization problem may be found in [7] among others and are not explicitly described herein.

Materials Research Proceedings **26** (2023) 405-410 https://doi.org/10.21741/9781644902431-66

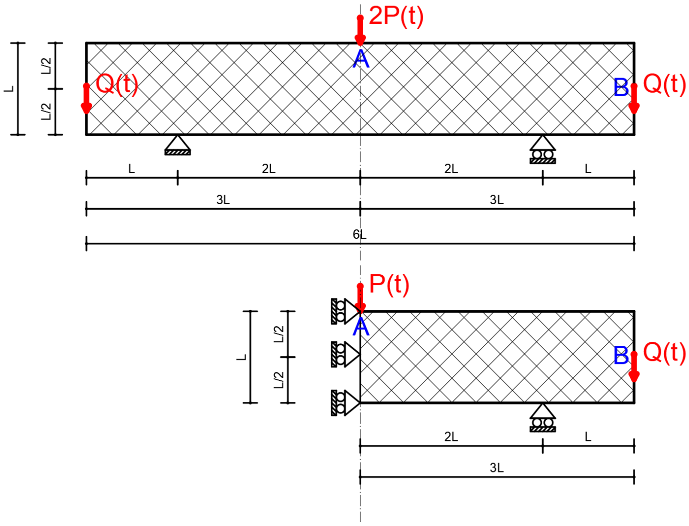

Figure 1 – The two-loaded beam: overall representation and symmetric problem

The problems under investigation. Three different optimization problems are considered and solved that are best defined referring to the so-called *descriptor state-space* formulation that reads:

$$\begin{cases} E\dot{x} = Ax + B\,u \\ y = Cx + D\,u \end{cases}, \tag{10}$$

where Eq. 10.1 is in fact the equation of motion in which x is the state vector that piles nodal displacements and velocities, u is the load vector, E is the *descriptor* (mass) matrix. A is the structural matrix that depends on the stiffness and damping matrices, and B is a topological matrix that distributes the loads to the degrees-of-freedom. Equation 10.2 is classically referred to as output equation and in fact y is the output vector that encompasses all the quantities that the optimization process should explicitly take care of (i.e. minimize). What is actually minimized is a suitable norm of the transfer-function matrix $G(s)$ that defines the Laplace-domain input/output relation $Y(s) = G(s)U(s)$ that may be shown to be equivalent to Eq. 10 and is graphically interpreted by the block in Fig. 2.

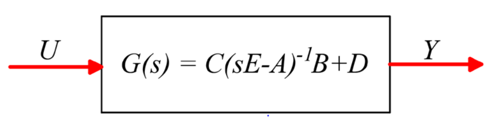

Figure 2 – Input-output Laplace-domain version of the system dynamics.

Given the system dynamics of Eq. 10.1, the specific optimization problem is fully defined by:

- The choice of the output vector y through a proper selection of the matrices C and D. At this regard both single-output and multi-output systems may inherently be handled by the proposed formulation. It should be noted that both single-output and multi-output cases give rise to a scalar minimization problem since a suitable norm of the resulting transfer function is the actual goal function that is minimized. Alternative approaches of vector minimization in a Pareto framework are left to future investigations.
- The selection of a proper matrix norm that is expected to address from a system-theoretic point of view the engineering goals that are expected to be reached by the designer.

Given this general scenario, the following optimization problems are considered.

1. Dynamic compliance H_∞*-norm* minimization (SI-SO). The two loads $P(t)$ and $Q(t)$ are supposed to belong to the same load combination (Single-Input), whereas the output is the so-called dynamic compliance [3];
2. The two loads $P(t)$, $Q(t)$ and the dual displacements $u_v(A)$, $u_v(B)$ define a 2-inputs, 2-outputs (MI-MO) transfer function of which the H_∞*-norm* is minimized;
3. Same as previous Problem 2 (MI-MO) but for the choice of the system norm to be minimized. A weighted version of the nuclear norm is chosen, i.e. $a_1\sigma_1 + a_2\sigma_2$.

Main results are briefly showcased next.

Problem 1: dynamic compliance H_∞-norm minimization. Figure 3 shows the optimal topology that solves the classical dynamic compliance problem along with the maximum-singular-value vs frequency curve (that for SI-SO systems is the same as the amplitude of the frequency response). The value of the goal function at convergence is 7.36 dB that is attained for $\omega = 0.28$ rad/s.

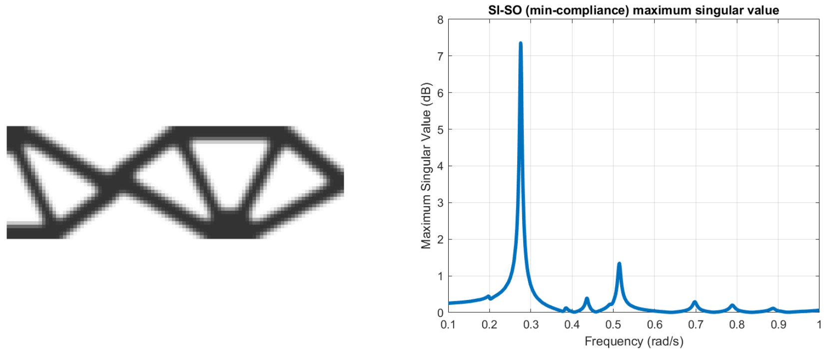

Figure 3 – Design case 1 – Optimal topology and maximum singular value vs frequency

Problem 2: H_∞-norm minimization of the 2×2 transfer function. Figure 4 shows the optimal topology that solves the MI-MO min H_∞-norm problem along with the maximum-singular-value vs frequency curve. The value of the goal function at convergence is 15.52 dB that is attained for $\omega = 0.048$ rad/s.

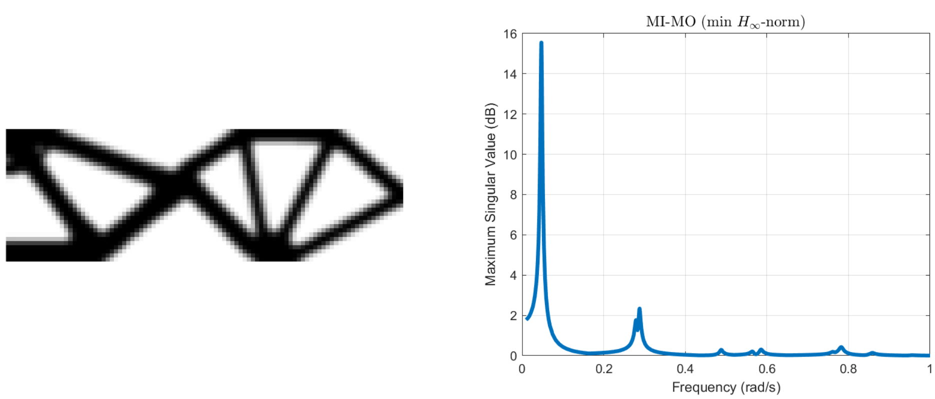

Figure 4 – Design case 2 – Optimal topology and maximum singular value

Problem 3: Nuclear-norm minimization of the 2×2 transfer function. Figure 5 shows the optimal topology that solves the MI-MO min $nuclear$ −norm problem along with the maximum-singular-value vs frequency curve. The value of the goal function at convergence is 53.87 dB that is attained for $\omega = 0.028$ rad/s.

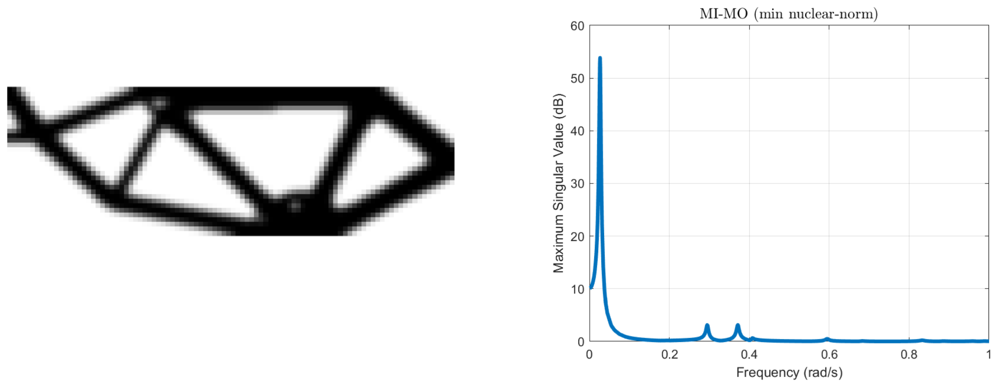

Figure 5 – Design case 3 – Optimal topology and maximum singular value

References

[1] J.S. Jensen, P.B. Nakshatrala, D.A. Tortorelli, On the consistency of adjoint sensitivity analysis for structural optimization of linear dynamic problems, Struct. Multidisc. Optim. 49 (2014) 831-837. https://doi.org/10.1007/s00158-013-1024-4

[2] N. Olhoff, J. Du, Generalized incremental frequency method for topological design of continuum structures for minimum dynamic compliance subject to forced vibration at a prescribed low or high value of the excitation frequency, Struct. Multidisc. Optim. 54 (2016) 1113-1141. https://doi.org/10.1007/s00158-016-1574-3

[3] P. Venini, Topology optimization of dynamic systems uncer uncertain loads: an H_∞-norm-based approach, J. Comput. Nonlinear Dynam. 14 (2019) 021007. https://doi.org/10.1115/1.4042140

[4] J. Du, N. Olhoff, Topological design of freely vibrating continuum structures for maximum values of simple and multiple eigenfrequencies and frequency gaps, Struct. Multidisc. Optim. 34 (2007) 91-110. https://doi.org/10.1007/s00158-007-0101-y

[5] G. Strang, Linear algebra and learning from data, Wellesley – Cambridge Press, Wellesley, 2019.

[6] S.L. Brunton, J.N. Kutz, Data-driven science and engineering – Machine learning, dynamical systems and control, Cambrige University Press, Cambridge, 2019. https://doi.org/10.1017/9781108380690

[7] P. Venini, P. Ceresa, A rational H_∞-norm-based approach for the optimal design of of sesmically excited reinforced concrete frames, Earthquake Engng Struct. Dyn. 47 (2018) 1522-1543. https://doi.org/10.1002/eqe.3028