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Lower bound limit analysis through discontinuous finite elements and semi-analytical procedures

Zona Renato[1,a *], Esposito Luca[1,b], Ferla Paolo[1,c], Palladino Simone[1,d], Totaro Elena[1,e], and Vincenzo Minutolo[1,f]

[1]Department of Engineering, University of Campania "Luigi Vanvitelli", Aversa (CE), Italy

[a]renato.zona@unicampania.it, [b]esposito.luca@unicampania.it, [c]paolo.ferla@unicampania.it, [d]simone.palladino@unicampania.it, [e]elena.totaro@unicampania.it, [f]vincenzo.minutolo@unicampania.it

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Abstract. This work deals with the limit analysis of structures through the lower-bound theorem, using dislocations based finite elements and eigenstress modelling. The lower bound approach is based on the knowledge of the self-equilibrated stresses that constitutes the basis of the domain where the optimal solution should be searched. A twofold strategy can be used to get self-equilibrated stresses, i.e., eigenstresses. The first one pursues the calculation of the self-equilibrated stress through the numerical approximation of the differential equilibrium equation in homogeneous form through an *a posteriori* discretization that used polynomial representation of finite degree. The second one consists of Finite Element implementation of the self-equilibrated stress calculation by discontinuous finite elements based on Volterra's dislocations theory. Both the formulations are written in terms of the strain and precisely in terms of the strain nodal displacement parameters. Consequently, it is possible to formulate an iterative procedure starting from the knowledge of the dislocation at the incoming collapse, in Melan's residual sense, and calculate the structural ductility requirement. Several numerical examples are presented to confirm the method's feasibility.

Introduction

The application of limit analysis and plasticity to structural safety concerns a wide range of engineering fields. Starting from the pioneer works of Prager, Drucker, and Greenberg's [1,2] and Massonnet and Save [3,4] that address the plastic response of structures introducing the collapse calculation for one-dimensional beams assembly as the main topic the matter has been formalized in the mathematical treatise of Hill [5], a two-fold approach is the way the limit analysis has been applied. The first, the kinematic method, consists of finding the collapse load as the load infimum among that in equilibrium with the stress linked to a compatible collapse mechanism. The second, the static approach, is based on the research of the supremum of the load in equilibrium with an admissible stress state that is a combination of self-equilibrated stress with the particular solution of the elastic equilibrium equation to the applied loads. An extensive chapter in the limit analysis is devoted to the mechanics of masonry structures following the work of Heyman [6], who extends the primary approach of limit analysis to the not tensile resistant materials (NTRM) modeling masonry. The limit analysis, and in a more general sense the plasticity modeling, is highly required when one must interpret the results of monitoring campaign since the structures during their life generally undergo permanent strain and cracks [7,8]. In the field of the biomechanics, the limit equilibrium is used as a primary tool to assess the fracture risk of prosthesis implants and relative optimization strategies as reported in [9,10,11].

The main topic of the limit analysis when addressed using the lower-bound approach is to define the field of the self-equilibrated stress that constitute the kernel of the equilibrium operator. The way the self-equilibrated stress is obtained characterizes the work here presented. Namely, a first strategy is based on mixed numerical-analytical solution of the equilibrium differential equations. The procedure evaluates the collapse load multiplier for masonry domes and vaults [12,13] and concrete caps. The results from the proposed formulation showed a good agreement with the experiments reported in [14]and with calculation presented in [15] that uses a kinematic approach based on the energy balance about crack lines which constitutes a typical pattern of collapse for plates and slender domes under radial load [16].
Alternatively, one can resolve the self-equilibrated stress, and consequently the domain within one must define the admissible stress state, with reference to finite element discretization of the model. The second procedure relates the permanent strain modeled as Volterra's dislocation to nodal parameters analogous to finite element nodal displacement. The formulated displacement base FEM gives the linear operator that relates the self-equilibrated stress domain to nodal parameters and boundary displacements. The dimensions of the self-equilibrated stress domain is the rank of the linear operator that in the case of truss and frames structures coincide with the redundancy degree of the structure.

Semi analytical method (SAM)

The first procedure consists of the search of the collapse multiplier through a mixed numerical and analytical procedure. The equilibrium equation for vaults and domes is solved through an optimization constrained problem. We have used a generalized stress formulation; hence the stress has been described through the internal forces, N, T, and M that are the resultant components, axial, and shear forces respectively, and the resultant bending moment of the stress acting on the section. Where the subscript $\{\ \}_{1,2}$ refers to the meridian or parallel cross section of the structure.

$$\begin{aligned} &\frac{d(N_1 r)}{d\theta} - N_2 R_1 \cos\theta - T_1 r = -X\, R_1 r \\ &N_1 r + N_2 R_1 \sin\theta + \frac{d(T_1 r)}{d\theta} = Z\, R_1 r \\ &\frac{d(M_1 r)}{d\theta} - M_2 R_1 \cos\theta - T_1\, R_1 r = 0 \end{aligned} \tag{1}$$

The set of equilibrium equation is solved numerically starting from a set of shape function and collocating the equation at the discrete colatitude angle θ_j .

$$\begin{cases} N_1^j = K_{N_1}^j c \\ N_2^j = K_{N_2}^j c \\ M_1^j = K_{M_1}^j c \\ M_2^j = K_{M_2}^j c \\ T_1^j = K_{T_1}^j c \end{cases} \tag{2}$$

The limit multiplier of prescribed load paths, either monotonically increasing or randomly variable, is obtained by maximizing the static multiplier of loads under the constraint that the sum of elastic response, plus any self-equilibrated time-independent stress solution, belongs to the admissible domain. Hence it results that the optimization program has the load multiplier as objective function and the parameters c as design variables. The optimization constraints are the linear inequalities representing the limit domain in terms of c. The elastic solution under the actual loads must be obtained from the equilibrium equation in any way. Namely, if closed form solution exists one can use it or can be obtained employing FEM analysis. The vectors N_i^e, M_i^e collected the effective generalized stress. Collocating the equations at discrete angles, i.e., at a finite number of θ_j with $j \in \{1, \dots \dots, m\}$, where m was the number of points along the meridian curve, one gets the desired solution. Finally, the optimization program, has the following discretized form:

$$\sup_{\mathbf{c}} k \left| k \in \mathbb{R}^+ : \begin{cases} h(N_i^r + kN_i^e) + \alpha(M_i^r + kM_i^e) < \beta h \\ h(N_i^r + kN_i^e) - \alpha(M_i^r + kM_i^e) < \beta h \\ (N_i^r + kN_i^e) < \beta \end{cases} \right|^j \tag{3}$$

Where β accounts for the presence of tensile resistance σ_0 such that axial limit stress is $N_0 = \sigma_0 h$ following the material constitutive properties.

Discontinuous finite element procedure (DFEP)

An alternative procedure uses finite elements in a discontinuous form to write the discrete operator that relates the set of self-equilibrated stress in the structure to discrete nodal displacement-like parameters. The approach is devoted to relating the permanent strain equivalent to Volterra's dislocation to nodal parameters. Moreover, the permanent strain and the corresponding self-equilibrated stress can be represented in terms of constraints displacements too. The representation can be assumed as a span of the eigenstress space. It can be seen that the span is not independent and that the base of the eigenstress is a proper subset of the parameters manifold.

In conclusion, any equilibrated stress under prescribed load can be calculated as the sum of the elastic response plus self-equilibrated stress, depending on the application of the linear operator V that maps the nodal dislocation parameter to the eigenstress.

The load collapse multiplier results as the 'sup' of the load multiplier in the constrained optimization program

$$s_\alpha = \sup k | f(k\sigma^* + V\delta) \leq 0, \alpha = \begin{cases} sd & \text{shakedown} \\ c & \text{collapse} \end{cases} \tag{4}$$

where σ^* is the stress in the structure calculated as being indefinitely elastic at any time during the load path, δ is the nodal dislocation parameter vector.

Results

Slender concrete caps have been analyzed following the *SAM* procedure and compared with experimental results obtained by [14,17] are reported in the following Fig.1 and Fig.2.

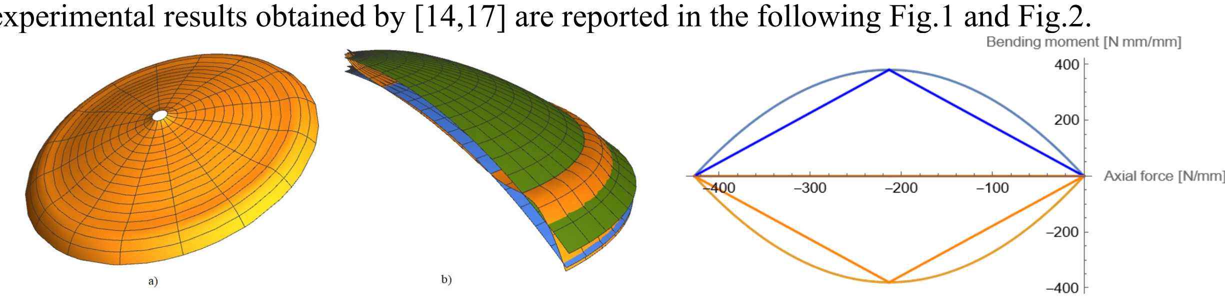

Figure 1 Thrust Surface and limit domain

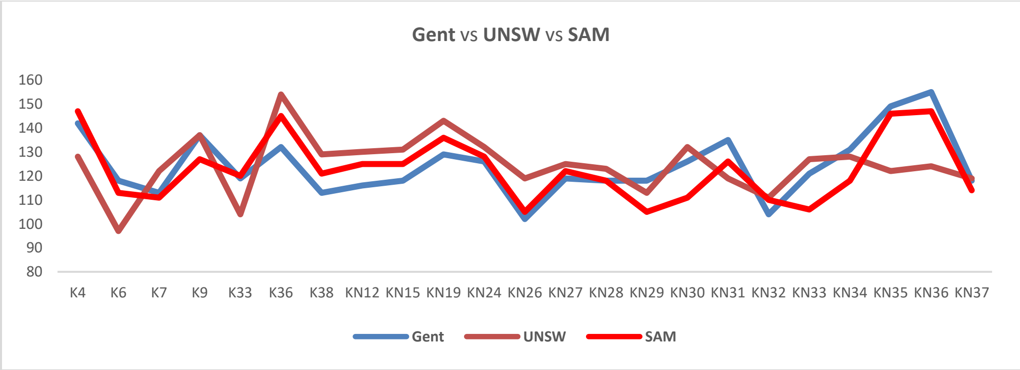

Figure 1. Result comparison among experimental results (Gent), analytical solution (UNSW) and proposed method (SAM) for different speciement $\boldsymbol{K_i}$

The *DFEP* has been applied to two-dimensional and three-dimensional examples. The first represent simple approximation of the Prestwood bridge Figure.3, whose collapse has been experimented in [18].

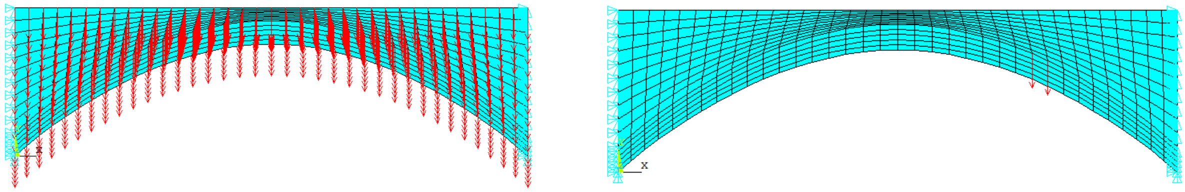

Figure 3 Bridge load condition

In Tab.1 the results for each mesh sizing are reported. The value of experimental load multiplier, in the destructive test conducted by Page is *228* KN.

Table 1 Load Multiplier values for each mesh size

Case	Element number along X	Element number along Y	Load Multiplier
1	18	5	701
2	36	5	423
3	72	5	312
4	72	10	258
5	88	12	**233**

The 3D example has modeled the plastic behavior of the cross section of a femur [9], modeled as a hollow cylinder, Fig.4, , after a hip prosthesis implant.The limit load multiplier has been compared with Ansys nonlinear analysis solution.

Table 2 Result comparison of femur limit load

Ansys Nonlinear solution	DFEP solution
13.64 MPa	12.86 Mpa

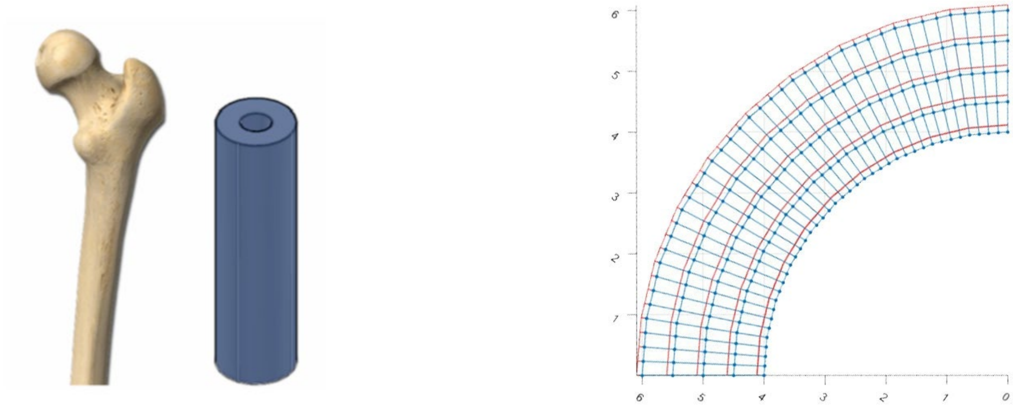

Figure 4 Approximation of the femur to a cave cylinder and elastic solution of the generic section (blue, undeformed shape/ red, deformed shape)

Both the examples confirm the accuracy of the procedure.

Conclusions

Two procedures have been set up. The first one, *SAM*, starts with the analytical solution of spherical domes to calculate any self-equilibrated stress. Moreover, a finite element elastic solution is obtained from the actual loads and was used as the purely elastic solution introduced into Melan's theorem. The analytical solution of the homogeneous equation of spherical domes is used to model the eigenstress of parabolical, conic, and slender domes through an approximate interpolation of the parabola with a sphere. The approximation allowed us to use the analytical solution to different non-spheric geometries. The proposed results were presented in terms of thrust lines. The results have been compared with numerical results obtained through commercial software of numerical analysis and experimental results. The thrust lines confirmed that the analyzed domes are safe under the applied loads or confirmed the collapse load multiplier under prescribed loads. The proposed method allowed us to calculate the safety factor under prescribed load patterns and assess the safety of the prescribed load level. Both presented strategies have shown the feasibility of the methods.

In second one, *DFEP*, the problem's schematization does not depend on its size.

The advantages of the procedure are manifold. First, the method allows the implementation of the load conditions, and the value of the collapse multiplier is computed without necessarily following the load path. Furthermore, being a FEM-based procedure, it can be borrowed from other commercial computing platforms for numerical analysis.

It should be noted that the *DFEP* procedure is structured in the deformation space. This fact has to be intended as a passage to an optimization problem in the displacement field, having chosen both the basic parameters of the self-stresses and the compatibility domain of the stresses in the displacement space. Therefore, in the proposed solution strategy, it is possible to control *a posteriori* the demand for structural ductility, which is essential for permanent deformations to unfold up to the desired load level without affecting the results obtained with a fragile local collapse.

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