Materials Research Proceedings **21** (2022) 384-390 https://doi.org/10.21741/9781644901755-65

Nonlinear Mathematical Models of Metamaterials Defined as "Mass-in-Mass" and "Damper-in-Mass" Chains

Daniil A. Kolesov[1,a*], Vladimir I. Erofeev[1,b], Alexey O. Malkhanov[1,c] and Ashot V. Shekoyan[2,d]

[1]Mechanical Engineering Research Institute of the Russian Academy of Sciences – Branch of Federal Research Center "Institute of Applied Physics of the RAS" ,85, Belinsky Street, Nizhny Novgorod, 603024, Russia

[2]Institute of Mechanics of the National Academy of Sciences of the Republic of Armenia, 24B, Baghramyan Ave., Yerevan, 0019, Republic of Armenia

[a]kolesov.d.a@yandex.ru, [b]erof.vi@yandex.ru, [c]alexey.malkhanov@gmail.com, [d]ashotshek@mechins.sci.am

Keywords: Metamaterial, Chains of Elastic and Viscoelastic Elements, Nonlinearity, Deformation

Abstract. To describe dynamic processes in an acoustic (mechanical) metamaterial, there are proposed models that are a one-dimensional chain containing the same masses connected by linearly elastic (or nonlinearly elastic) elements (springs) with the same stiffness. In this case, it is assumed that each mass contains inside itself a series connection of another mass and an elastic element or viscous element (damper).

Introduction

An adequate description of the physical and mechanical properties of metamaterials within the framework of the classical theory of elasticity is impossible. Recently, generalized micropolar theories of the Cosserat continuum type have become widespread for modeling structurally inhomogeneous materials [1]. However, these theories include a large number of material constants that require experimental determination and the connection of which with the structure of the material is not clear. An alternative direction, structural modeling, is devoid of such a drawback [2].

Nonlinear elastic "mass-in-mass" chain

In [3], to describe the dynamic properties of a metamaterial, a one-dimensional chain was considered containing the same masses m_1 connected by elastic elements (springs) with the same stiffness k_1, while each mass inside itself contained another mass m_2 and one more elastic element - a spring with stiffness k_2 (Fig. 1). This model is called a "mass-in-mass chain".

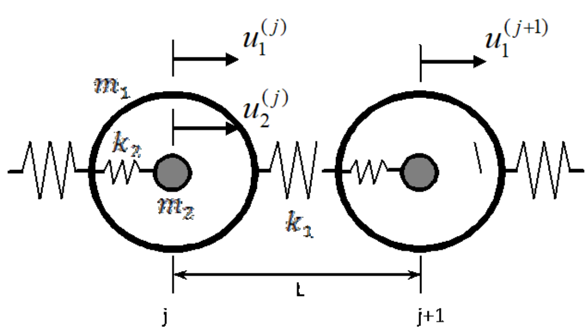

Fig. 1. *The mechanical model of elastic metamaterial*

Materials Research Proceedings 21 (2022) 384-390 https://doi.org/10.21741/9781644901755-65

Let us generalize the model [3] by taking into account the quadratic nonlinearity of the external and internal elastic elements.

The potential energy of the unit cell of the "mass-in-mass" chain will be written as:

$$W^{(j)} = \frac{1}{2}\Big[k_1\Big(u_1^{(j+1)} - u_1^{(j)}\Big)^2 + k_2\Big(u_2^{(j)} - u_1^{(j)}\Big)^2 + h_1\Big(u_1^{(j+1)} - u_1^{(j)}\Big)^3 + h_2\Big(u_2^{(j)} - u_1^{(j)}\Big)^3\Big], \tag{1}$$

and its kinetic energy in the form:

$$T^{(j)} = \frac{1}{2}\Big[m_1\Big(\dot{u}_1^{(j)}\Big)^2 + m_2\Big(\dot{u}_2^{(j)}\Big)^2\Big]. \tag{2}$$

Let us suppose that $u_1(x)$ and $u_2(x)$ are continuous functions that give offsets of all m_1 and m_2, respectively. Using the Taylor series expansion of displacements, and restricting ourselves to two terms, we obtain

$$u_1^{(j+1)} = u_1(x+L) = u_1(x) + \frac{\partial u_1}{\partial x}L = u_1^{(j)} + \frac{\partial u_1}{\partial x}L. \tag{3}$$

The displacement decomposition technique in (3) was effectively used by I.A. Kunin [4] in the transformation of multi-mass discrete systems into a quasi-continuum.

The potential and kinetic energy densities for the equivalent continuum obtained from (5) and (6) are written as:

$$W = \frac{1}{2L}\left[k_1\left(\frac{\partial u_1}{\partial x}L\right)^2 + k_2(u_2 - u_1)^2 + h_1\left(\frac{\partial u_1}{\partial x}L\right)^3 + h_2(u_2 - u_1)^3\right], \tag{4}$$

$$T = \frac{1}{2L}[m_1(\dot{u}_1)^2 + m_2(\dot{u}_2)^2]. \tag{5}$$

Let us compose from (4) and (5) the Lagrangian $\mathcal{L} = T - W = \mathcal{L}(\dot{u}_1, \dot{u}_2, u_{1x}, u_1, u_2)$ and use the equations of analytical mechanics

$$\frac{\partial}{\partial t}\left(\frac{d\mathcal{L}}{d\dot{u}_1}\right) + \frac{\partial}{\partial x}\left(\frac{\partial\mathcal{L}}{\partial u_{1x}}\right) - \frac{\partial\mathcal{L}}{\partial u_1} = 0,$$
$$\frac{\partial}{\partial t}\left(\frac{d\mathcal{L}}{d\dot{u}_2}\right) - \frac{\partial\mathcal{L}}{\partial u_2} = 0.$$

to obtain a system of equations in the displacements of equations (4), (5), we get:

$$\frac{m_1}{L}\ddot{u}_1 - k_1L\frac{\partial^2 u_1}{\partial x^2} - 3h_1L^3\frac{\partial u_1}{\partial x}\frac{\partial^2 u_1}{\partial x^2} - \frac{k_2}{L}(u_2 - u_1) - \frac{3h_2}{2L}(u_2 - u_1)^2 = 0,$$
$$\frac{m_1}{L}\ddot{u}_2 + \frac{k_2}{L}(u_2 - u_1) + \frac{3h_2}{2L}(u_2 - u_1)^2 = 0. \tag{6}$$

Further we will consider a particular case of system (6), where $h_1 \neq 0$, $h_2 = 0$, i.e.:

$$\frac{m_1}{L}\ddot{u}_1 - k_1 L\frac{\partial^2 u_1}{\partial x^2} - 3h_1 L^3\frac{\partial u_1}{\partial x}\frac{\partial^2 u_1}{\partial x^2} - \frac{k_2}{L}(u_2 - u_1) = 0,$$
$$\frac{m_2}{L}\ddot{u}_2 + \frac{k_2}{L}(u_2 - u_1) = 0. \tag{7}$$

System (7) can be rewritten as a single equation:

$$\frac{\partial^2 u_2}{\partial t^2} - \frac{k_1 L^2}{m_1+m_2}\frac{\partial^2 u_2}{\partial x^2} + \frac{m_1 m_2}{k_2(m_1+m_2)}\frac{\partial^4 u_2}{\partial t^4} - \frac{k_1 L^2 m_2}{k_2(m_1+m_2)}\frac{\partial^4 u_2}{\partial x^2 \partial t^2} - \frac{3h_1 L^4}{m_1+m_2}\left(\frac{\partial u_2}{\partial x}\frac{\partial^2 u_2}{\partial x^2} + \frac{m_2}{k_2}\frac{\partial^2 u_2}{\partial x^2}\frac{\partial^3 u_2}{\partial t^2 \partial x} + \frac{m_2}{k_2}\frac{\partial u_2}{\partial x}\frac{\partial^4 u_2}{\partial t^2 \partial x^2} + \left(\frac{m_2}{k_2}\right)^2\frac{\partial^3 u_2}{\partial t^2 \partial x}\frac{\partial^4 u_2}{\partial t^2 \partial x^2}\right) = 0. \tag{8}$$

Let's move on to dimensionless time, coordinate and displacement:

$$\tau = \frac{t}{T}, y = \frac{x}{X}, u_2 = u_0 u\,. \tag{9}$$

The transformed equation (8) with the substitutions (9) takes the following form:

$$\frac{\partial^2 u}{\partial \tau^2} - \frac{k_1 L^2}{m_1+m_2}\frac{T^2}{X^2}\frac{\partial^2 u}{\partial y^2} + \frac{m_1 m_2}{k_2(m_1+m_2)}\frac{1}{T^2}\frac{\partial^4 u}{\partial \tau^4} - \frac{k_1 L^2 m_2}{k_2(m_1+m_2)}\frac{1}{X^2}\frac{\partial^4 u}{\partial y^2 \partial \tau^2} - \frac{3h_1 L^4}{m_1+m_2}\frac{T^2 u_0}{X^3}\left(\frac{\partial u}{\partial y}\frac{\partial^2 u}{\partial y^2} + \frac{m_2}{k_2}\frac{1}{T^2}\frac{\partial^2 u}{\partial y^2}\frac{\partial^3 u}{\partial \tau^2 \partial y} + \frac{m_2}{k_2}\frac{1}{T^2}\frac{\partial u}{\partial y}\frac{\partial^4 u}{\partial \tau^2 \partial y^2} + \left(\frac{m_2}{k_2}\frac{1}{T^2}\right)^2\frac{\partial^3 u}{\partial \tau^2 \partial y}\frac{\partial^4 u}{\partial \tau^2 \partial y^2}\right) = 0. \tag{10}$$

We require that all coefficients (10) be finite or small values. Let us choose them so that among the nonlinear terms it is possible to single out only one, the main term.

All further discussions are valid under two conditions:

$$\frac{k_1 L^2}{m_1+m_2}\frac{T^2}{X^2} = 1, \frac{m_1 m_2}{k_2(m_1+m_2)}\frac{1}{T^2} = \varepsilon$$

where $\varepsilon << 1$.

When these conditions are met in equation (10), some of the terms can be neglected, since they have a higher order of smallness and do not have a significant effect on dynamic processes. Thus, equation (10) takes the form:

$$\frac{\partial^2 u}{\partial \tau^2} - \frac{\partial^2 u}{\partial y^2} + \varepsilon\frac{\partial^2}{\partial \tau^2}\left[\frac{\partial^2 u}{\partial \tau^2} - \alpha\frac{\partial^2 u}{\partial y^2}\right] = \delta\frac{\partial u}{\partial y}\frac{\partial^2 u}{\partial y^2}, \tag{11}$$

where $\frac{m_1 m_2}{k_2(m_1+m_2)}\frac{1}{T^2} = \varepsilon << 1, \frac{3h_1 L u_0\sqrt{\varepsilon}\alpha}{k_1\sqrt{\frac{k_1 m_2}{k_2 m_1}}} = \delta << 1, \alpha = \frac{m_1+m_2}{m_1} > 1.$

Returning to the original dimensional variables in equation (11), we obtain a simplified equation (8) in the form:

$$\frac{\partial^2 u_2}{\partial t^2}-\frac{k_1L^2}{m_1+m_2}\frac{\partial^2 u_2}{\partial x^2}+\frac{m_1m_2}{k_2(m_1+m_2)}\frac{\partial^4 u_2}{\partial t^4}-\frac{k_1L^2m_2}{k_2(m_1+m_2)}\frac{\partial^4 u_2}{\partial x^2\partial t^2}-\frac{3h_1L^4}{m_1+m_2}\frac{\partial u_2}{\partial x}\frac{\partial^2 u_2}{\partial x^2}=0. \tag{12}$$

Visco-elastic "damper-in-mass" chain

It is not possible to study the dissipative properties of a metamaterial within the framework of a purely elastic formulation of problem (12). To solve this problem, we replace the elastic element with stiffness k_2 by a viscous element (Fig. 2).

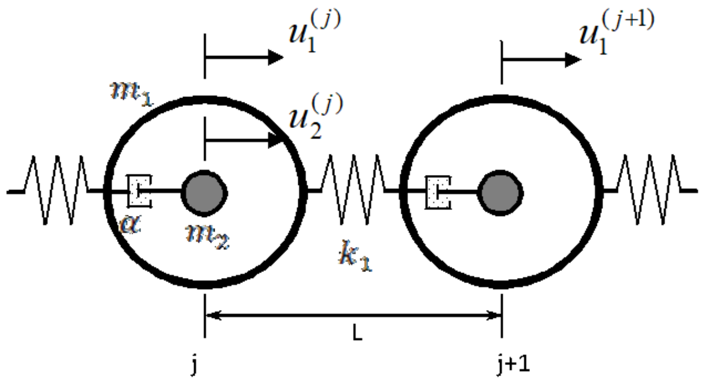

Fig. 2. *The mechanical model of a viscoelastic metamaterial*

The dynamics equations of the modified "mass-in-mass" chain in the long-wavelength range will have the form:

$$\frac{m_1}{L}\frac{\partial^2 u_1}{\partial t^2}-k_1L\frac{\partial^2 u_1}{\partial x^2}-\frac{\alpha}{L}\frac{\partial}{\partial t}(u_2-u_1)=0, \tag{13}$$

$$\frac{m_2}{L}\frac{\partial^2 u_2}{\partial t^2}+\frac{\alpha}{L}\frac{\partial}{\partial t}(u_2-u_1)=0. \tag{14}$$

Note that system (13), (14) can be reduced to one equation for displacement:

$$\frac{\partial^2 u_1}{\partial t^2}-C_0^2\frac{\partial^2 u_1}{\partial x^2}+\frac{m_1m_2}{L\alpha(m_1+m_2)}\frac{\partial^3 u_1}{\partial t^3}-\frac{C_0^2m_2}{L\alpha}\frac{\partial^3 u_1}{\partial x^2\partial t}=0. \tag{15}$$

If we introduce in (15) the dimensionless displacement $U=\frac{u_1}{u_0}$, coordinate $X=\frac{x}{a}$ and time $T=\frac{t}{b}$, where $b=\frac{m_2}{L\alpha}$, $a=C_0b$, then this equation can be rewritten as:

$$\frac{\partial^2 U}{\partial T^2}-\frac{\partial^2 U}{\partial X^2}+\delta\frac{\partial^3 U}{\partial T^3}-\frac{\partial^3 U}{\partial X^2\partial T}=0. \tag{16}$$

Here $\delta=\frac{m_1}{m_1+m_2}$. This parameter belongs to the interval δ = [0,1], which includes two limiting cases: $\delta\to 1$, if $m_1\gg m_2$ and $\delta\to 0$, if $m_2\gg m_1$.

We consider equation (16) with the following initial conditions:

$$U(X,0)=Asech(\gamma X)=\frac{2A}{e^{\gamma X}+e^{-\gamma X}}, \tag{17}$$

$$\frac{\partial U(X,0)}{\partial t} = 0, \tag{18}$$

where A – is an amplitude, γ – is a spatial parameter.

The development of the initial ($T_0 = 0$) disturbance (17), (18) can be traced over the next three time instants (Fig. 3).

The solution to the problem is symmetric with respect to $X = 0$, because the initial value (17) is an even function. Solutions obtained at $\delta = 0.05$, are plotted to the left of the axis of symmetry (dashed line), and the solutions obtained at $\delta = 0.5$, are plotted to the right of the axis.

Comparison of these cases shows the difference in dispersion. The character of the attenuation of disturbances can vary and depends on the value of δ. In the case of a small value of parameter δ the attenuation is much faster than when δ is greater. The initial sections (at $T = T_0$) in both cases are qualitatively similar. This is explained by the fact that in both cases anomalous dispersion takes place at large values of the wavenumber k. The main difference between the presented cases arises when considering the tail of the curves. At small values of parameter δ the solution behaves more like a solution of the diffusion equation, but for large values of parameter δ the solution behaves similarly to the solution of the wave equation. The presence of a more bulky terms at low values of parameter δ due to the superposition of the effects of normal dispersion and negative group velocity.

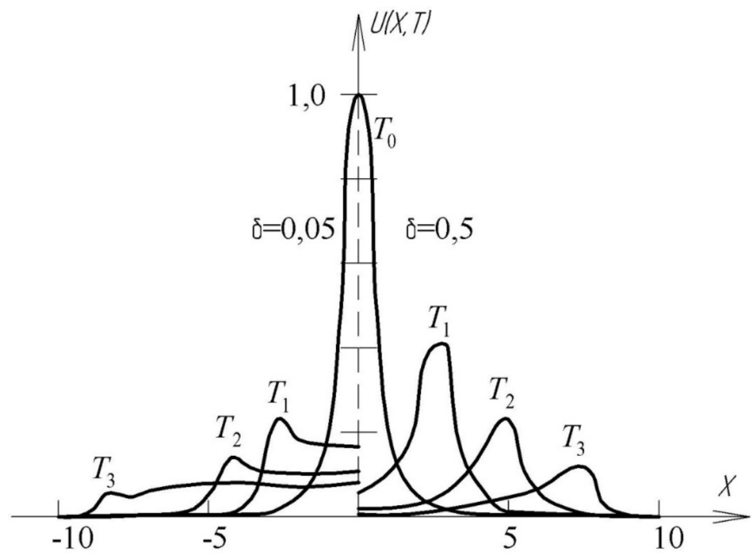

Fig. 3. *Instant wave profiles at* $A = 1$ *and* $\gamma = 3$*, at the moments* $T_0 = 0, T_1 = 7/3, T_2 = 14/3, T_3 = 21/3$*, which calculated for two values of parameter* δ*.*

In the Fig. 4 it is shown that the wave profiles for asymmetric development for four consecutive time instants. The initial value consists of the sum of two perturbations that have different fundamental frequencies. The initial perturbation has the form:

$$U(X,0) = Asech(\gamma X) + Bsech[g(X+1)], \tag{19}$$

$$\frac{\partial U(X,0)}{\partial t} = 0. \tag{20}$$

Here, the main perturbation, marked in Fig. 4 as a, has an amplitude A = 1 and the spatial parameter $\gamma = 1.6$ (corresponds to a disturbance with a low fundamental frequency). The secondary disturbance, which is marked in Fig. 4 as b, shifts to the left with a certain step with respect to the main disturbance. The secondary disturbance has an amplitude of $B = 0.55$ and a spatial parameter of $g = 10$, which corresponds to a high fundamental frequency.

In the Fig. 4 it is shown that the high-frequency perturbation b propagates faster than the main perturbation, which has a lower frequency. For example, in position b_2 and at the corresponding moment in time T_2, the maximum of perturbation b_0 reaches the maximum of the main perturbation, and in position b_3 the maximum of perturbation b_0 is ahead of the maximum of the main perturbation. This phenomenon is explained by anomalous dispersion, which is expressed in the fact that the group velocity exceeds the phase velocity.

From the results of dispersion analysis, it follows that the high-frequency wave components must also decay faster than the low-frequency components. Indeed, this statement is confirmed in Fig. 4. The peak value of the main perturbation decreased from the initial amplitude $A = 1$ at time T_0 to $A \approx 0.2$ at time T_3. On the other hand, the perturbation amplitude b_0 decreases more significantly, from $B = 0.55$ at time T_0 to $B \approx 0$ at time T_3.

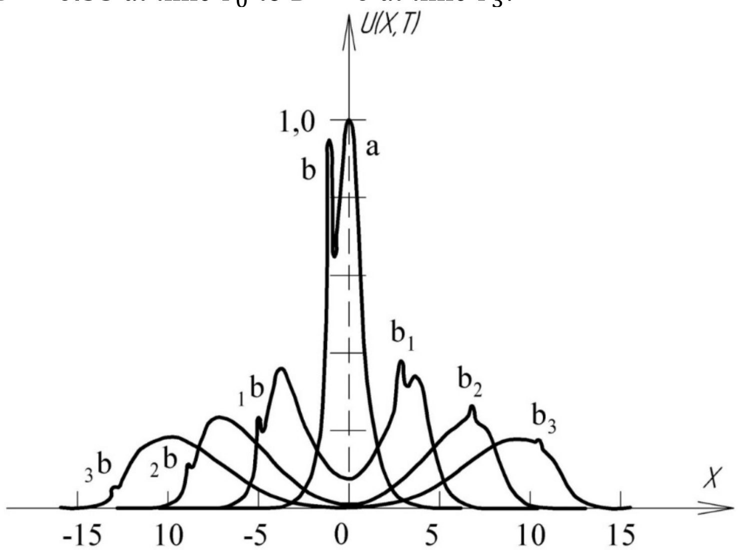

Fig. 4. *Instant wave profiles at* $\delta = 0.5, \gamma = 1.6, g = 10, A = 1, B = 0.55$, *at the moments* $T_0 = 0, T_1 = 15/4, T_2 = 30/4, T_3 = 45/4$. *Markers Маркеры* b_i *depict the place of the peak* b_0 *when the disturbance propagates the to right,* ${}_ib$ *– when the disturbance propagates the to left.*

As a result of the studies, it was shown that the longitudinal wave in the viscoelastic metamaterial, defined as the mass-in-mass chain, has dispersion and frequency-dependent attenuation. The evolution of the wave profile is analyzed, both in the low-frequency and in the high-frequency ranges.

Summary

The models "mass-in-mass" and "damper-in-mass" constructed in this work are free from the drawbacks inherent in a number of other mechanical models of metamaterials: they free from the need to endow deformable bodies with negative mass, density and (or) negative modulus of elasticity.

Acknowledgments
The research was carried out under the financial support of the Russian Scientific Foundation (Project No 21-19-00813).

References

[1] I. Vardoulakis, Cosserat Continuum Mechanics., Springer, Cham, Switzerland, 2019. https://doi.org/10.1007/978-3-319-95156-0

[2] V.I. Erofeev, I.S. Pavlov, Structural Modeling of Metamaterials, Springer Nature Switzerland, 2021. https://doi.org/10.1007/978-3-030-60330-4

[3] I.A. Kunin, Elastic Media with Microstructure. 1: One-Dimensional Models, Springer, Berlin, 1982. https://doi.org/10.1007/978-3-642-81748-9_1

[4] H.H. Huang, C.T. Sun, G.L. Huang, On the negative effective mass density in acoustic metamaterials. Int. J. Eng. Sci. 2009. Vol. 47. P.610-617. https://doi.org/10.1016/j.ijengsci.2008.12.007